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Calculus 2 for Engineering

التفاضل والتكامل للهندسة (2)

(MTH1182T)

(2025-2026)

الإسم : .....

الشعبة: .....

جـامعـــة أم الـــــقرى
كلية العلوم التطبيقية
قسم العلوم الرياضية

رمـز المقـــرر: MTH1182T
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المتطلب السابق:

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الخطة الدراسية لمقرر التفاضل والتكامل للهندسة (2) : MTH1182T

Weeks	Chapters	Sections Numbers	Topics
1		**Review of the Fundamental Theorems of Calculus**	
1	**Transcendental Functions** Calculus, 9th Edition, Purcell	**6.1**	**The Natural Logarithm Function** Examples: 1, 2, 3, 7, and 8. Exercises: 3, 5, 7, 9, 11, 13, 31, 33, 35, 37, and 38.
2		**6.2**	**Inverse Functions and Their Derivatives** Examples: *1, 2, 3, and 4.* Exercises: *7, 9, 11, 15, 17, 19, 21 and 23.*
2		**6.3**	**The Natural Exponential Function** Examples: 1, 2, and 3. Exercises: 3, 5, 11, 13, 15, 17, 19, 21, 27 and 29.
3		**6.4**	**General Exponential and Logarithmic Functions** Examples: 1, 2, 4, 5, 6, and 7. Exercises: 18, 20, 22, 28, 30, and.34.
3		**6.8**	**The Inverse Trigonometric Functions and Their Derivatives** Examples: 1, 3, 4, and 5. Exercises: 2, 39, 41, 44, 46, 48, 50, 52, and 54.
4		**6.9**	**The Hyperbolic Functions and Their Inverses** Examples: 1, 2, and 4. Exercises: 13, 15, 18, 20, 22, 24, 28, 30, 32, and 35.
4	**Techniques of Integration** Calculus, 9th Edition, Purcell	**7.1**	**Basic integration rules** Examples: 1, 2, 3, 4, 5, 6, 7 and 8. Exercises: 1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 27 and 29.
5		**7.2**	**Integration by parts** Examples: 1, 2, 3, 4, 5 and 6. Exercises: 2, 4, 5, 6, 13, 15, 19, 23, 27, 31, 35, 38, 42, and 45.
5		**7.3**	**Some trigonometric integrals** Examples: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11 Exercises: **1,** 3, 5, 8, 11, 13, 17, 21, 24, 26, and 28.
6		**7.4**	**Rationalizing substitutions** Examples: 1, 2, 3, 4, 5, 6, and 7. Exercises: **2,** 4, 6, 9, 11, 13, 15, 19, 22, 24, and 25.
6		**7.5**	**Integration of rational functions using partial fractions** Examples: **1, 2,** 3, 4, 5, 6, and 7. Exercises:, 2, 4, 6 and 30.
7	**Applications of the integral** Calculus, 9th Edition, Purcell	**5.1**	**The area of a plane region** Examples: 1, 2, 3, 4, 5 and 6. Exercises: 1, 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 21, and 29.
7		**5.2**	**Volumes of solids: Slabs, disks, washers** Examples: 1, 2, 3, 4, and 5. Exercises: 1, 3, 5, 7, 9, 11, 13, and 15.

8		**5.3**	**Volumes of solids of revolutions: Shells** Examples: 1, 2, 3, and 4. Exercises: 1, 3, 5, 7, 9, 11, 15, and 17.
		5.4	**Length of a plane curve** Examples: 1, 3, 4, 5, and 6. Exercises: 1, 3, 5, 7, 9, 13, and 15.
9	**Midterm Exam**		
10	**Intermediate Forms and Improper Integrals** Calculus, 9th Edition, Purcell	**8.1**	**Indeterminate Forms of Type 0/0** Examples: *1, 2, 3, 4, 5, and 6.* Exercises: *1, 3, 5, 7, 15, 17, 18, 20 and 22.*
		8.2	**Other Indeterminate Forms** Examples: *1, 2, 3, 4, 5, 6, 7 and 8.* Exercises: 1, 3, 5, 7, 11, 13, *15, 17, 19, and 21***.**
11		**8.3**	**Improper Integrals: Infinite limits of integration** Examples: 1, 2, 4, 6, and 7. Exercises: **1,** 3, 5, 8, 11, 13, 15, 17, 19, and 20
		8.4	**Improper Integrals: Infinite Integrands** Examples: 1, 2, 3, 4, 5, 6, and 7. Exercises: **2,** 4, 6, 9, 11, 13, 15, 19, 22, 24, 27, and 30**.**
12	**Infinite Series** Calculus, 9th Edition, Purcell	**9.1**	**Infinite Sequences** Examples: *1, 2, 3, 4, and 5.* Exercises: 1, 3, 4, 7, and 9**.**
		9.2	**Infinite Series** Examples: 1, 2, 3, 5, 6, 7, and 8. Exercises: 1, 2, 3, and 4**.**
13		**9.3**	**Positive series: The integral test** Examples: 1, 2, 3, and 5 Exercises: 1, 2, 3, and 5**.**
14		**9.4**	**Positive series: Other Tests** Examples: 1, 2, 5, 6, and 7. Exercises: 1, 3, and 4**.**
15		**9.5**	**Alternative Series, Absolute Convergence, and Conditional Convergence** Examples: *1, 2, 3, 4, and 6.* Exercises: 1, 2, 4, 7, 9, 11, 13, 15, 19, 21, and 29.

References:

Calculus (9th Edition), Dale Varberg, Edwin Purcell and Steven Rigdon, Prentice Hall (2006).2-

CHAPTER 6 Transcendental Functions

6.1 The Natural Logarithm Function
6.2 Inverse Functions and Their Derivatives
6.3 The Natural Exponential Function
6.4 General Exponential and Logarithmic Functions
6.5 Exponential Growth and Decay
6.6 First-Order Linear Differential Equations
6.7 Approximations for Differential Equations
6.8 The Inverse Trigonometric Functions and Their Derivatives
6.9 The Hyperbolic Functions and Their Inverses

6.1 The Natural Logarithm Function

The power of calculus, both that of derivatives and integrals, has already been amply demonstrated. Yet we have only scratched the surface of potential applications. To dig deeper, we need to expand the class of functions with which we can work. That is the object of this chapter.

We begin by asking you to notice a peculiar gap in our knowledge of derivatives.

$$D_x\left(\frac{x^2}{2}\right) = x^1,\ D_x(x) = x^0,\ D_x(??) = x^{-1},\ D_x\left(-\frac{1}{x}\right) = x^{-2},\ D_x\left(-\frac{x^{-2}}{2}\right) = x^{-3}$$

Is there a function whose derivative is $1/x$? In other words, is there an antiderivative $\int 1/x\,dx$? The First Fundamental Theorem of Calculus states that the accumulation function

$$F(x) = \int_a^x f(t)\,dt$$

is a function whose derivative is $f(x)$, provided that f is continuous on an interval I that contains a and x. In this sense, we can find an antiderivative of *any* continuous function. The existence of an antiderivative does not mean that the antiderivative can be expressed in terms of functions that we have studied so far. In this chapter we will introduce and study a number of new functions.

Our first new function is chosen to fill the gap noticed above. We call it the **natural logarithm function,** and it does have something to do with the logarithm studied in algebra, but this relationship will only appear later. For now, just accept the fact that we are going to define a *new* function and study its properties.

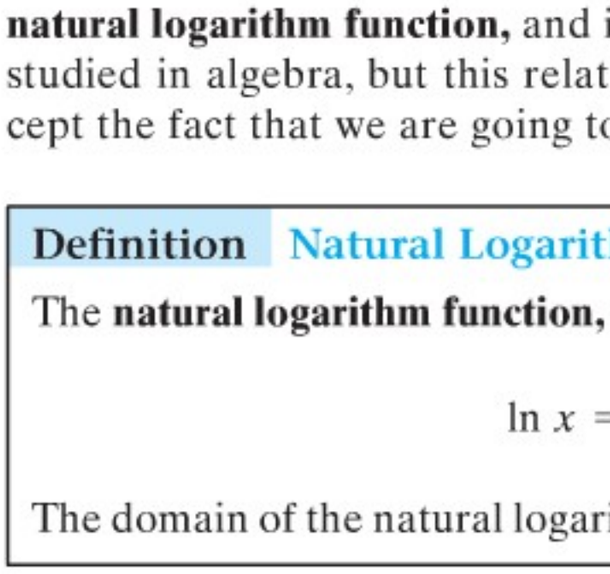
Definition Natural Logarithm Function

The **natural logarithm function,** denoted by ln, is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

The domain of the natural logarithm function is the set of positive real numbers.

The diagrams in Figure 1 indicate the geometric meaning of $\ln x$. The natural logarithm (or natural log) function measures the area under the curve $y = 1/t$ between 1 and x if $x > 1$ and the negative of the area if $0 < x < 1$. The natural logarithm is an accumulation function because it accumulates area under the curve



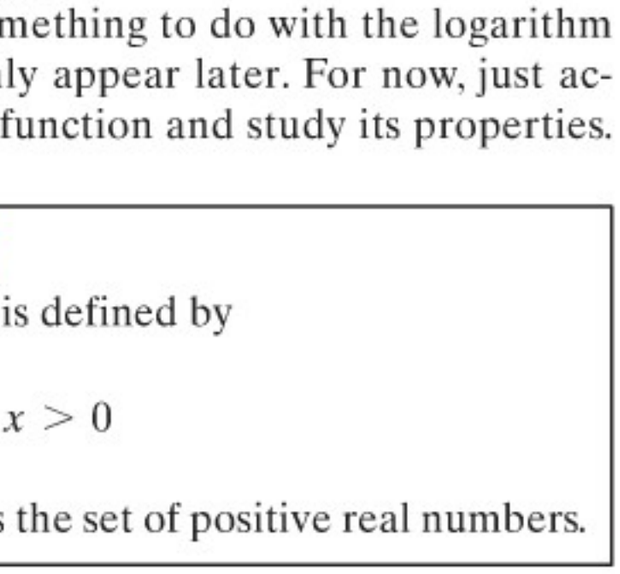


Figure 1

$y = 1/t$. Clearly, $\ln x$ is well defined for $x > 0$; $\ln x$ is not defined for $x \leq 0$ because this definite integral does not exist over an interval that includes 0.

And what is the derivative of this new function? Just exactly what we want.

Derivative of the Natural Logarithm Function From the First Fundamental Theorem of Calculus, we have

$$D_x \int_1^x \frac{1}{t}\,dt = D_x \ln x = \frac{1}{x}, \qquad x > 0$$

This can be combined with the Chain Rule. If $u = f(x) > 0$ and if f is differentiable, then

$$D_x \ln u = \frac{1}{u} D_x u$$

EXAMPLE 1 Find $D_x \ln \sqrt{x}$.

SOLUTION Let $u = \sqrt{x} = x^{1/2}$. Then

$$D_x \ln \sqrt{x} = \frac{1}{x^{1/2}} \cdot D_x(x^{1/2}) = \frac{1}{x^{1/2}} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2x}$$ ■

EXAMPLE 2 Find $D_x \ln(x^2 - x - 2)$.

SOLUTION This problem makes sense, provided that $x^2 - x - 2 > 0$. Now $x^2 - x - 2 = (x - 2)(x + 1)$, which is positive provided that $x < -1$ or $x > 2$. Thus, the domain of $\ln(x^2 - x - 2)$ is $(-\infty, -1) \cup (2, \infty)$. On this domain,

$$D_x \ln(x^2 - x - 2) = \frac{1}{x^2 - x - 2} D_x(x^2 - x - 2) = \frac{2x - 1}{x^2 - x - 2}$$ ■

EXAMPLE 3 Show that

$$D_x \ln|x| = \frac{1}{x}, \qquad x \neq 0$$

SOLUTION Two cases are to be considered. If $x > 0$, $|x| = x$, and

$$D_x \ln|x| = D_x \ln x = \frac{1}{x}$$

If $x < 0$, $|x| = -x$, and so

$$D_x \ln|x| = D_x \ln(-x) = \frac{1}{-x} D_x(-x) = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}$$ ■

We know that for every differentiation formula there is a corresponding integration formula. Thus, Example 3 implies that

$$\int \frac{1}{x}\,dx = \ln|x| + C, \qquad x \neq 0$$

or, with u replacing x,

$$\int \frac{1}{u}\,du = \ln|u| + C, \qquad u \neq 0$$

This fills the long-standing gap in the Power Rule: $\int u^r\,du = u^{r+1}/(r + 1)$, from which we had to exclude the exponent $r = -1$.

EXAMPLE 4 Find $\int \frac{5}{2x+7}\,dx$.

SOLUTION Let $u = 2x + 7$, so $du = 2\,dx$. Then

$$\int \frac{5}{2x+7}\,dx = \frac{5}{2}\int \frac{1}{2x+7}\,2\,dx = \frac{5}{2}\int \frac{1}{u}\,du$$

$$= \frac{5}{2}\ln|u| + C = \frac{5}{2}\ln|2x+7| + C$$ ■

EXAMPLE 5 Evaluate $\int_{-1}^{3} \frac{x}{10 - x^2}\,dx$.

SOLUTION Let $u = 10 - x^2$, so $du = -2x\,dx$. Then

$$\int \frac{x}{10-x^2}\,dx = -\frac{1}{2}\int \frac{-2x}{10-x^2}\,dx = -\frac{1}{2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|10 - x^2| + C$$

Thus, by the Second Fundamental Theorem of Calculus,

$$\int_{-1}^{3} \frac{x}{10-x^2}\,dx = \left[-\frac{1}{2}\ln|10-x^2|\right]_{-1}^{3} = -\frac{1}{2}\ln 1 + \frac{1}{2}\ln 9 = \frac{1}{2}\ln 9$$

For the above calculation to be valid, $10 - x^2$ must never be 0 on the interval $[-1, 3]$. It is easy to see that this is true. ■

When the integrand is the quotient of two polynomials (that is, a rational function) and the numerator is of equal or greater degree than the denominator, always *divide the denominator into the numerator first.*

EXAMPLE 6 Find $\int \frac{x^2 - x}{x + 1}\,dx$.

SOLUTION By long division (Figure 2),

$$\frac{x^2 - x}{x+1} = x - 2 + \frac{2}{x+1}$$

Hence,

$$\int \frac{x^2-x}{x+1}\,dx = \int (x-2)\,dx + 2\int \frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2} - 2x + 2\int \frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2} - 2x + 2\ln|x+1| + C$$ ■

$$\begin{array}{r} x-2 \\ x+1 \overline{\big)\, x^2 - x} \\ \underline{x^2 + x} \\ -2x \\ \underline{-2x - 2} \\ 2 \end{array}$$

Figure 2

Common Logarithms

Properties (ii) and (iii) for **common logarithms** (base 10 logarithms) were what motivated the invention of logarithms. John Napier (1550–1617) wanted to simplify the complicated calculations that arose in astronomy and navigation. To replace multiplication by addition and division by subtraction was his goal—exactly what (ii) and (iii) accomplish. For over 350 years, common logarithms were an essential aid in computation, but today we use calculators and computers for this purpose. However, natural logarithms retain their importance for other reasons, as you will see.

Properties of the Natural Logarithm The next theorem lists several important properties of the natural log function.

Theorem A

If a and b are positive numbers and r is any rational number, then

(i) $\ln 1 = 0$;

(ii) $\ln ab = \ln a + \ln b$;

(iii) $\ln \frac{a}{b} = \ln a - \ln b$;

(iv) $\ln a^r = r \ln a$.

Proof

(i) $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$

(ii) Since, for $x > 0$,

$$D_x \ln ax = \frac{1}{ax} \cdot a = \frac{1}{x}$$

and

$$D_x \ln x = \frac{1}{x}$$

it follows from the theorem about two functions with the same derivative (Theorem 3.6B) that

$$\ln ax = \ln x + C$$

To determine C, let $x = 1$, obtaining $\ln a = C$. Thus,

$$\ln ax = \ln x + \ln a$$

Finally, let $x = b$.

(iii) Replace a by $1/b$ in (ii) to obtain

$$\ln\frac{1}{b} + \ln b = \ln\left(\frac{1}{b} \cdot b\right) = \ln 1 = 0$$

Thus,

$$\ln\frac{1}{b} = -\ln b$$

Applying (ii) again, we get

$$\ln\frac{a}{b} = \ln\left(a \cdot \frac{1}{b}\right) = \ln a + \ln\frac{1}{b} = \ln a - \ln b$$

(iv) Since, for $x > 0$,

$$D_x(\ln x^r) = \frac{1}{x^r} \cdot r x^{r-1} = \frac{r}{x}$$

and

$$D_x(r \ln x) = r \cdot \frac{1}{x} = \frac{r}{x}$$

it follows by Theorem 3.6B, which we used in (ii), that

$$\ln x^r = r \ln x + C$$

Let $x = 1$, which gives $C = 0$. Thus,

$$\ln x^r = r \ln x$$

Finally, let $x = a$. ■

EXAMPLE 7 Find dy/dx if $y = \ln \sqrt[3]{(x-1)/x^2}$, $x > 1$.

SOLUTION Our task is easier if we first use the properties of natural logarithms to simplify y.

$$y = \ln\left(\frac{x-1}{x^2}\right)^{1/3} = \frac{1}{3}\ln\left(\frac{x-1}{x^2}\right)$$

$$= \frac{1}{3}\left[\ln(x-1) - \ln x^2\right] = \frac{1}{3}\left[\ln(x-1) - 2\ln x\right]$$

Thus,

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x-1} - \frac{2}{x}\right] = \frac{2-x}{3x(x-1)}$$

■

Logarithmic Differentiation The labor of differentiating expressions involving quotients, products, or powers can often be substantially reduced by first applying the natural logarithm function and using its properties. This method, called **logarithmic differentiation,** is illustrated in Example 8.

EXAMPLE 8 Differentiate $y = \dfrac{\sqrt{1-x^2}}{(x+1)^{2/3}}$.

SOLUTION First we take natural logarithms; then we differentiate implicitly with respect to x (recall Section 2.7).

$$\ln y = \frac{1}{2}\ln(1-x^2) - \frac{2}{3}\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-2x}{2(1-x^2)} - \frac{2}{3(x+1)} = \frac{-(x+2)}{3(1-x^2)}$$

Thus,

$$\frac{dy}{dx} = \frac{-y(x+2)}{3(1-x^2)} = \frac{-\sqrt{1-x^2}\,(x+2)}{3(x+1)^{2/3}(1-x^2)}$$

$$= \frac{-(x+2)}{3(x+1)^{2/3}(1-x^2)^{1/2}}$$

■

Example 8 could have been done directly, without first taking logarithms, and we suggest you try it. You should be able to make the two answers agree.

The Graph of the Natural Logarithm The domain of $\ln x$ consists of the set of all positive real numbers, so the graph of $y = \ln x$ is in the right half-plane. Also, for $x > 0$,

$$D_x \ln x = \frac{1}{x} > 0$$

and

$$D_x^2 \ln x = -\frac{1}{x^2} < 0$$

The first formula tells us that the natural log function is continuous (why?) and rises as x increases; the second tells us that the graph is everywhere concave down. In Problems 43 and 44, you are asked to show that

$$\lim_{x\to\infty} \ln x = \infty$$

and

$$\lim_{x\to 0^+} \ln x = -\infty$$

Finally, $\ln 1 = 0$. These facts imply that the graph of $y = \ln x$ is similar in shape to that shown in Figure 3.

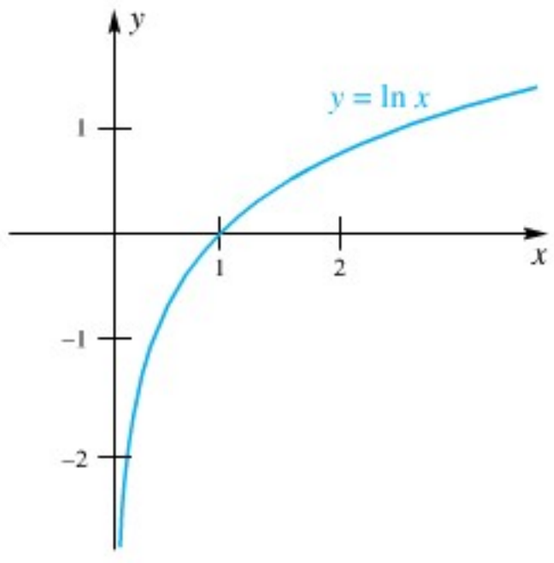


Figure 3

If your calculator has an [ln] button, values for the natural logarithm are at your fingertips. For example,

$$\ln 2 \approx 0.6931$$

$$\ln 3 \approx 1.0986$$

Trigonometric Integrals Some trigonometric integrals can be evaluated using the natural log function.

EXAMPLE 9 Evaluate $\displaystyle\int \tan x\,dx$.

SOLUTION Since $\tan x = \dfrac{\sin x}{\cos x}$ we can make the substitution $u = \cos x$, $du = -\sin x\,dx$, to obtain

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-1}{\cos x}(-\sin x\,dx) = -\ln|\cos x| + C \quad \blacksquare$$

Similarly, $\displaystyle\int \cot x\,dx = \ln|\sin x|$.

EXAMPLE 10 Evaluate $\displaystyle\int \sec x \csc x\,dx$.

SOLUTION For this one we use the trig identity $\sec x \csc x = \tan x + \cot x$. Then

$$\int \sec x \csc x\,dx = \int (\tan x + \cot x)\,dx = -\ln|\cos x| + \ln|\sin x| + C \quad \blacksquare$$

Concepts Review

1. The function ln is defined by $\ln x =$ _____. The domain of this function is _____ and its range is _____.

2. From the preceding definition, it follows that $D_x \ln x =$ _____ for $x > 0$.

3. More generally, for $x \neq 0$, $D_x \ln|x| =$ _____ and so $\int (1/x)\,dx =$ _____.

4. Some common properties of ln are $\ln(xy) =$ _____, $\ln(x/y) =$ _____, and $\ln(x^r) =$ _____.

Problem Set 6.1

1. Use the approximations $\ln 2 \approx 0.693$ and $\ln 3 \approx 1.099$ together with the properties stated in Theorem A to calculate approximations to each of the following. For example, $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3 \approx 0.693 + 1.099 = 1.792$.

(a) $\ln 6$ (b) $\ln 1.5$ (c) $\ln 81$

(d) $\ln \sqrt{2}$ (e) $\ln\left(\frac{1}{36}\right)$ (f) $\ln 48$

2. Use your calculator to make the computations in Problem 1 directly.

In Problems 3–14, find the indicated derivative (see Examples 1 and 2). Assume in each case that x is restricted so that ln is defined.

3. $D_x \ln(x^2 + 3x + \pi)$

4. $D_x \ln(3x^3 + 2x)$

5. $D_x \ln(x - 4)^3$

6. $D_x \ln \sqrt{3x - 2}$

7. $\dfrac{dy}{dx}$ if $y = 3 \ln x$

8. $\dfrac{dy}{dx}$ if $y = x^2 \ln x$

9. $\dfrac{dz}{dx}$ if $z = x^2 \ln x^2 + (\ln x)^3$

10. $\dfrac{dr}{dx}$ if $r = \dfrac{\ln x}{x^2 \ln x^2} + \left(\ln \dfrac{1}{x}\right)^3$

11. $g'(x)$ if $g(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$

12. $h'(x)$ if $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $f'(81)$ if $f(x) = \ln \sqrt[3]{x}$

14. $f'\left(\dfrac{\pi}{4}\right)$ if $f(x) = \ln(\cos x)$

In Problems 15–26, find the integrals (see Examples 4, 5, and 6).

15. $\displaystyle\int \frac{1}{2x + 1}\,dx$

16. $\displaystyle\int \frac{1}{1 - 2x}\,dx$

17. $\displaystyle\int \frac{6v + 9}{3v^2 + 9v}\,dv$

18. $\displaystyle\int \frac{z}{2z^2 + 8}\,dz$

19. $\displaystyle\int \frac{2 \ln x}{x}\,dx$

20. $\displaystyle\int \frac{-1}{x(\ln x)^2}\,dx$

21. $\displaystyle\int_0^3 \frac{x^4}{2x^5 + \pi}\,dx$

22. $\displaystyle\int_0^1 \frac{t + 1}{2t^2 + 4t + 3}\,dt$

23. $\displaystyle\int \frac{x^2}{x - 1}\,dx$

24. $\displaystyle\int \frac{x^2 + x}{2x - 1}\,dx$

25. $\displaystyle\int \frac{x^4}{x + 4}\,dx$

26. $\displaystyle\int \frac{x^3 + x^2}{x + 2}\,dx$

In Problems 27–30, use Theorem A to write the expressions as the logarithm of a single quantity.

27. $2 \ln(x + 1) - \ln x$

28. $\frac{1}{2}\ln(x - 9) + \frac{1}{2}\ln x$

29. $\ln(x - 2) - \ln(x + 2) + 2 \ln x$

30. $\ln(x^2 - 9) - 2 \ln(x - 3) - \ln(x + 3)$

In Problems 31–34, find dy/dx by logarithmic differentiation (see Example 8).

31. $y = \dfrac{x + 11}{\sqrt{x^3 - 4}}$

32. $y = (x^2 + 3x)(x - 2)(x^2 + 1)$

33. $y = \dfrac{\sqrt{x + 13}}{(x - 4)\sqrt[3]{2x + 1}}$

34. $y = \dfrac{(x^2 + 3)^{2/3}(3x + 2)^2}{\sqrt{x + 1}}$

In Problems 35–38, make use of the known graph of $y = \ln x$ to sketch the graphs of the equations.

35. $y = \ln|x|$

36. $y = \ln \sqrt{x}$

37. $y = \ln\left(\dfrac{1}{x}\right)$

38. $y = \ln(x - 2)$

39. Sketch the graph of $y = \ln \cos x + \ln \sec x$ on $(-\pi/2, \pi/2)$, but think before you begin.

40. Explain why $\lim_{x \to 0} \ln \dfrac{\sin x}{x} = 0$.

41. Find all local extreme values of $f(x) = 2x^2 \ln x - x^2$ on its domain.

42. The rate of transmission in a telegraph cable is observed to be proportional to $x^2 \ln(1/x)$, where x is the ratio of the radius of the core to the thickness of the insulation $(0 < x < 1)$. What value of x gives the maximum rate of transmission?

43. Use the fact that $\ln 4 > 1$ to show that $\ln 4^m > m$ for $m > 1$. Conclude that $\ln x$ can be made as large as desired by choosing x sufficiently large. What does this imply about $\lim_{x \to \infty} \ln x$?

44. Use the fact that $\ln x = -\ln(1/x)$ and Problem 43 to show that $\lim_{x \to 0^+} \ln x = -\infty$.

45. Solve for x: $\displaystyle\int_{1/3}^{x} \frac{1}{t}\,dt = 2\int_{1}^{x} \frac{1}{t}\,dt$.

46. Prove the following statements.

(a) Since $1/t < 1/\sqrt{t}$ for $t > 1$, $\ln x < 2\left(\sqrt{x} - 1\right)$ for $x > 1$.

(b) $\lim_{x \to \infty} (\ln x)/x = 0$.

47. Calculate

$$\lim_{n \to \infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right]$$

by writing the expression in brackets as

$$\left[\frac{1}{1 + 1/n} + \frac{1}{1 + 2/n} + \cdots + \frac{1}{1 + n/n}\right]\frac{1}{n}$$

and recognizing the latter as a Riemann sum.

C **48.** A famous theorem (the Prime Number Theorem) says that the number of primes less than n for large n is approximately $n/(\ln n)$. About how many primes are there less than 1,000,000?

49. Find and simplify $f'(1)$.

(a) $f(x) = \ln\left(\dfrac{ax - b}{ax + b}\right)^c$, where $c = \dfrac{a^2 - b^2}{2ab}$.

(b) $f(x) = \displaystyle\int_1^u \cos^2 t\,dt$, where $u = \ln(x^2 + x - 1)$.

50. Evaluate $\displaystyle\int_0^{\pi/3} \tan x\,dx$.

51. Evaluate $\displaystyle\int_{\pi/4}^{\pi/3} \sec x \csc x\,dx$.

52. Evaluate $\displaystyle\int \frac{\cos x}{1 + \sin x}\,dx$.

53. The region bounded by $y = (x^2 + 4)^{-1}$, $y = 0$, $x = 1$, and $x = 4$, is revolved about the y-axis, generating a solid. Find its volume.

54. Find the length of the curve $y = x^2/4 - \ln \sqrt{x}$, $1 \le x \le 2$.

55. By appealing to the graph of $y = 1/x$, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

56. Prove **Napier's Inequality,** which says that, for $0 < x < y$,

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}$$

CAS **57.** Let $f(x) = \ln(1.5 + \sin x)$.

(a) Find the absolute extreme points on $[0, 3\pi]$.

(b) Find any inflection points on $[0, 3\pi]$.

(c) Evaluate $\displaystyle\int_0^{3\pi} \ln(1.5 + \sin x)\,dx$.

CAS **58.** Let $f(x) = \cos(\ln x)$.

(a) Find the absolute extreme points on $[0.1, 20]$.

(b) Find the absolute extreme points on $[0.01, 20]$.

(c) Evaluate $\displaystyle\int_{0.1}^{20} \cos(\ln x)\,dx$.

CAS **59.** Draw the graphs of $f(x) = x \ln(1/x)$ and $g(x) = x^2 \ln(1/x)$ on $(0, 1]$.

(a) Find the area of the region between these curves on $(0, 1]$.

(b) Find the absolute maximum value of $|f(x) - g(x)|$ on $(0, 1]$.

CAS **60.** Follow the directions of Problem 59 for $f(x) = x \ln x$ and $g(x) = \sqrt{x} \ln x$.

Answers to Concepts Review: **1.** $\displaystyle\int_1^x (1/t)\,dt$; $(0, \infty)$; $(-\infty, \infty)$ **2.** $1/x$ **3.** $1/x$; $\ln|x| + C$ **4.** $\ln x + \ln y$; $\ln x - \ln y$; $r \ln x$

6.2 Inverse Functions and Their Derivatives

Our stated aim for this chapter is to expand the number of functions in our repertoire. One way to manufacture new functions is to take old ones and "reverse" them. When we do this for the natural logarithm function, we will be led to the natural exponential function, the subject of Section 6.3. In this section, we study the general problem of reversing (or inverting) a function. Here is the idea.

A function f takes a number x from its domain D and assigns to it a single value y from its range R. If we are lucky, as in the case of the two functions

graphed in Figures 1 and 2, we can reverse f; that is, for any given y in R, we can unambiguously go back and find the x from which it came. This new function that takes y and assigns x to it is denoted by f^{-1}. Note that its domain is R and its range is D. It is called the **inverse** of f, or simply f-inverse. Here we are using the superscript -1 in a new way. The symbol f^{-1} does not denote $1/f$, as you might have expected. We, and all mathematicians, use it to name the inverse function.

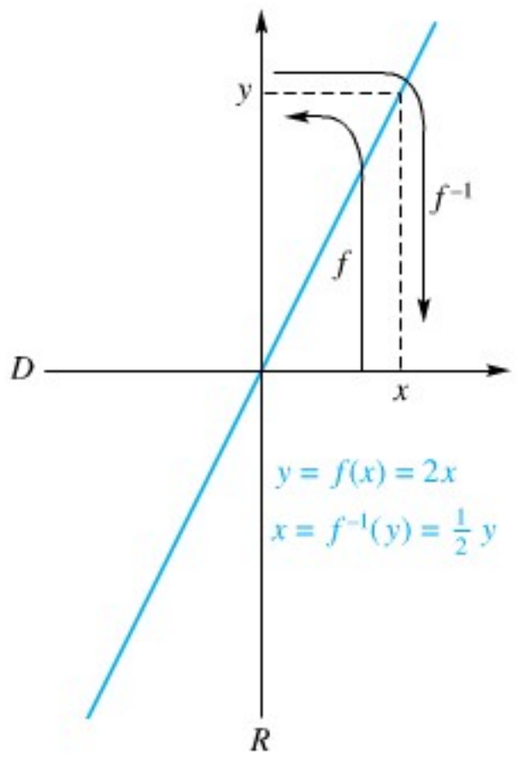


Figure 1

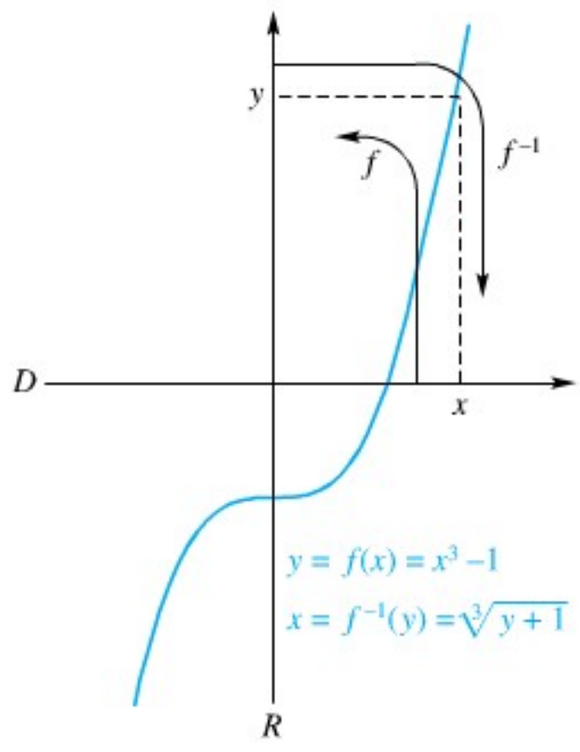


Figure 2

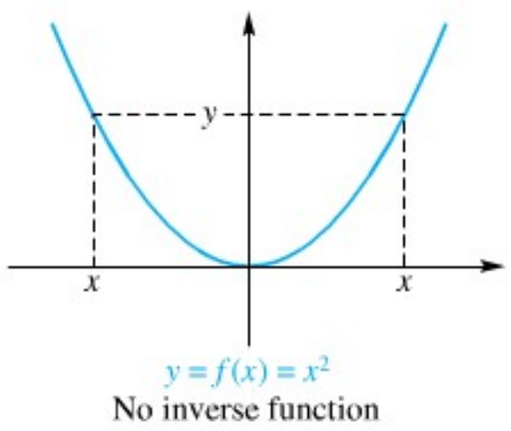


Figure 3

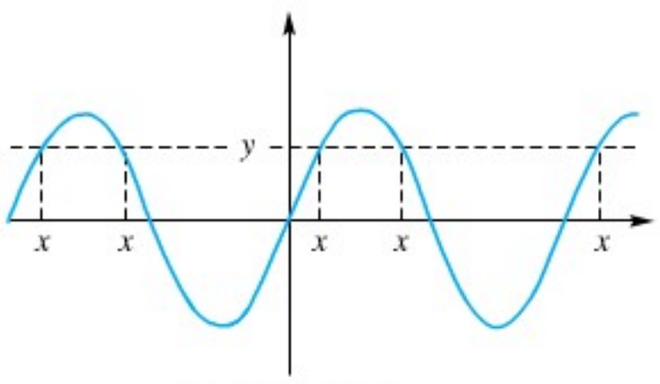


Figure 4

Sometimes, we can give a formula for f^{-1}. If $y = f(x) = 2x$, then $x = f^{-1}(y) = \frac{1}{2}y$ (see Figure 1). Similarly, if $y = f(x) = x^3 - 1$, then $x = f^{-1}(y) = \sqrt[3]{y + 1}$ (Figure 2). In each case, we simply solve the equation that determines f for x in terms of y. The result is $x = f^{-1}(y)$.

But life is more complicated than these two examples indicate. Not every function can be reversed in an unambiguous way. Consider $y = f(x) = x^2$, for example. For a given y there are *two* x's that correspond to it (Figure 3). The function $y = g(x) = \sin x$ is even worse. For each y, there are infinitely many x's that correspond to it (Figure 4). Such functions do not have inverses; at least, they do not unless we somehow restrict the set of x-values, a subject we will take up later.

Existence of Inverse Functions It would be nice to have a simple criterion for deciding whether a function f has an inverse. One such criterion is that the function be **one-to-one;** that is, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. This is equivalent to the geometric condition that every horizontal line meet the graph of $y = f(x)$ in at most one point. But, in a given situation, this criterion may be very hard to apply, since it demands that we have complete knowledge of the graph. A more practical criterion that covers most examples that arise in this book is that a function be **strictly monotonic.** By this we mean that it is either increasing or decreasing on its domain. (See the definitions in Section 3.2.)

Theorem A

If f is strictly monotonic on its domain, then f has an inverse.

Proof Let x_1 and x_2 be distinct numbers in the domain of f, with $x_1 < x_2$. Since f is monotonic, $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$. Either way, $f(x_1) \neq f(x_2)$. Thus, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, which means that f is one-to-one and therefore has an inverse. ■

This is a practical result, because we have an easy way of deciding whether a differentiable function f is strictly monotonic. We simply examine the sign of f'.

EXAMPLE 1 Show that $f(x) = x^5 + 2x + 1$ has an inverse.

SOLUTION $f'(x) = 5x^4 + 2 > 0$ for all x. Thus, f is increasing on the whole real line and so it has an inverse there. ■

We do not claim that we can always give a formula for f^{-1}. In the example just considered, this would require that we be able to solve $y = x^5 + 2x + 1$ for x. Although we could use a CAS or a graphing calculator to solve this equation for x for a particular value of y, there is no simple formula that would give us x in terms of y for an arbitrary y.

There is a way of salvaging the notion of inverse for functions that do not have inverses on their natural domain. We simply *restrict the domain* to a set on which the graph is either increasing or decreasing. Thus, for $y = f(x) = x^2$, we may restrict the domain to $x \geq 0$ ($x \leq 0$ would also work). For $y = g(x) = \sin x$, we restrict the domain to the interval $[-\pi/2, \pi/2]$. Then both functions have inverses (see Figure 5), and we can even give a formula for the first one: $f^{-1}(y) = \sqrt{y}$.

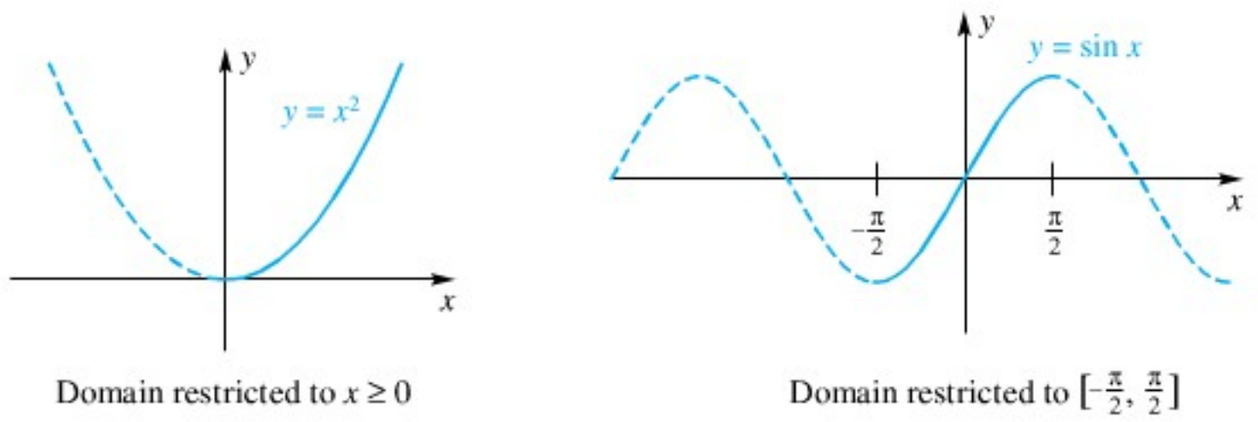


Figure 5

If f has an inverse f^{-1}, then f^{-1} also has an inverse, namely, f. Thus, we may call f and f^{-1} a pair of inverse functions. One function undoes (or reverses) what the other did; that is,

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

EXAMPLE 2 Show that $f(x) = 2x + 6$ has an inverse, find a formula for $f^{-1}(y)$, and verify the results in the box above.

SOLUTION Since f is an increasing function, it has an inverse. To find $f^{-1}(y)$, we solve $y = 2x + 6$ for x, which gives $x = (y - 6)/2 = f^{-1}(y)$. Finally, note that

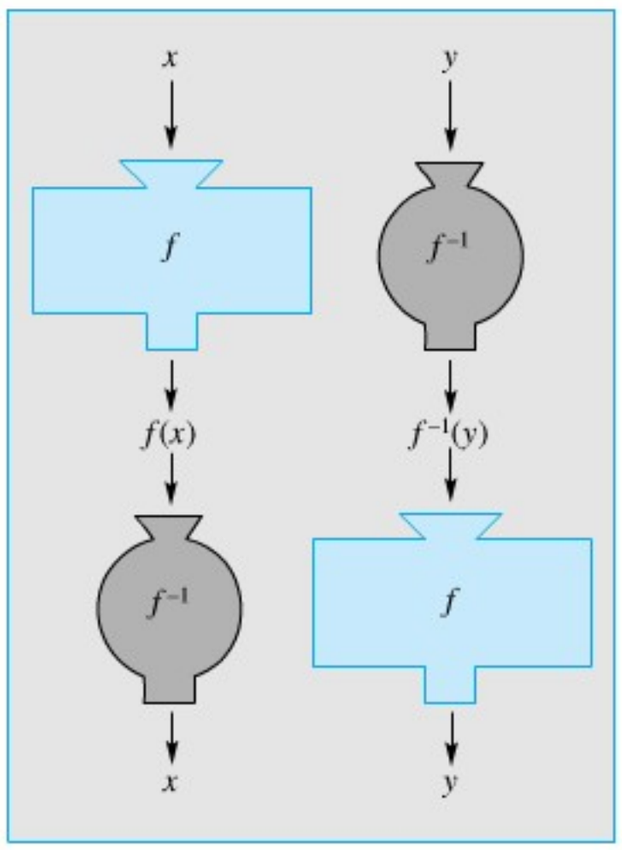
$$f^{-1}(f(x)) = f^{-1}(2x + 6) = \frac{(2x + 6) - 6}{2} = x$$

and

$$f(f^{-1}(y)) = f\left(\frac{y - 6}{2}\right) = 2\left(\frac{y - 6}{2}\right) + 6 = y$$ ■

Undoing Machines

We may view a function as a machine that accepts an input and produces an output. If the f machine and the f^{-1} machine are hooked together in tandem, they undo each other.



The Graph of $y = f^{-1}(x)$ Suppose that f has an inverse. Then

$$x = f^{-1}(y) \Leftrightarrow y = f(x)$$

Consequently, $y = f(x)$ and $x = f^{-1}(y)$ determine the same (x, y) pairs and so have identical graphs. However, it is conventional to use x as the domain variable for functions, so we now inquire about the graph of $y = f^{-1}(x)$ (note that we have interchanged the roles of x and y). A little thought convinces us that to interchange the roles of x and y on a graph is to reflect the graph across the line $y = x$.

Thus the graph of $y = f^{-1}(x)$ *is just the reflection of the graph of* $y = f(x)$ *across the line* $y = x$ (Figure 6).

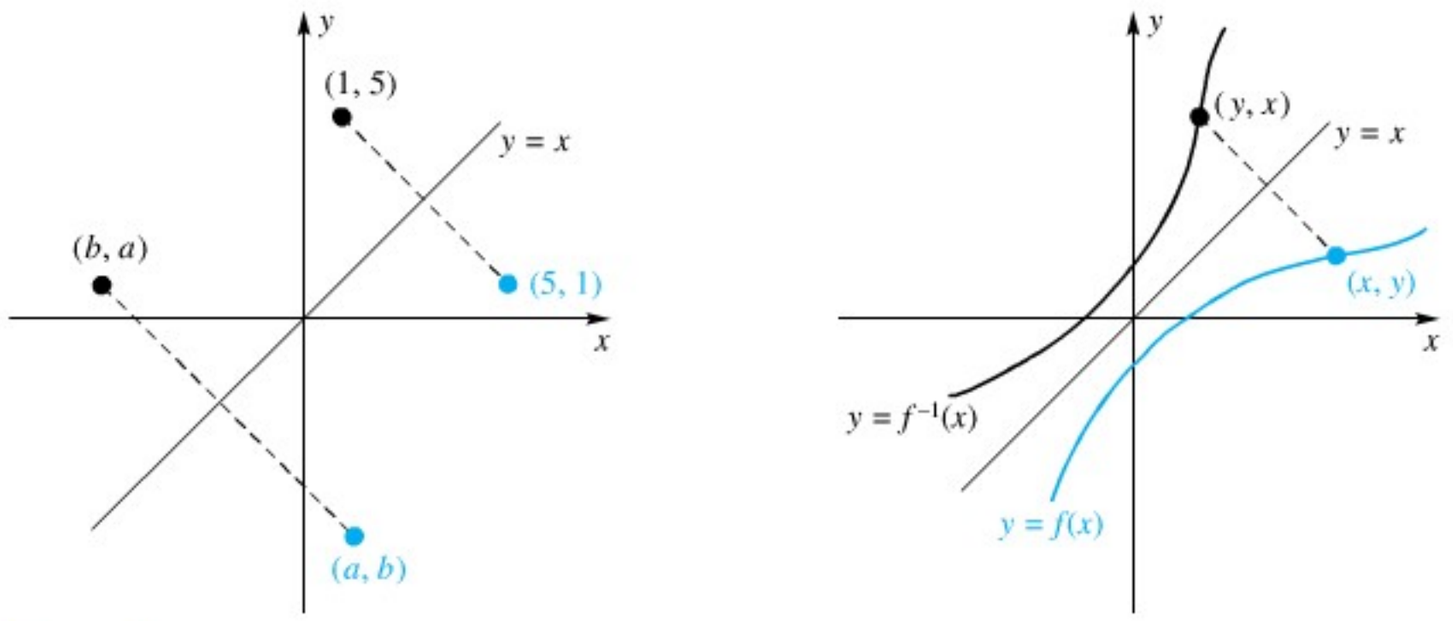


Figure 6

A related matter is that of finding a formula for $f^{-1}(x)$. To do it, we first find $f^{-1}(y)$ and then replace y by x in the resulting formula. Thus, we propose the following three-step process for finding $f^{-1}(x)$.

Step 1: Solve the equation $y = f(x)$ for x in terms of y.

Step 2: Use $f^{-1}(y)$ to name the resulting expression in y.

Step 3: Replace y by x to get the formula for $f^{-1}(x)$.

Before trying the three-step process on a particular function f, you might think we should first verify that f has an inverse. However, if we can actually carry out the first step and get a single x for each y, then f^{-1} does exist. (Note that when we try this for $y = f(x) = x^2$ we get $x = \pm\sqrt{y}$, which immediately shows that f^{-1} does not exist, unless, of course, we have restricted the domain to eliminate one of the two signs, + or −.)

EXAMPLE 3 Find a formula for $f^{-1}(x)$ if $y = f(x) = x/(1 - x)$.

SOLUTION Here are the three steps for this example.

Step 1:

$$y = \frac{x}{1 - x}$$

$$(1 - x)y = x$$

$$y - xy = x$$

$$x + xy = y$$

$$x(1 + y) = y$$

$$x = \frac{y}{1 + y}$$

Step 2: $f^{-1}(y) = \dfrac{y}{1 + y}$

Step 3: $f^{-1}(x) = \dfrac{x}{1 + x}$ ■

Derivatives of Inverse Functions We conclude this section by investigating the relationship between the derivative of a function and the derivative of its inverse. Consider first what happens to a line l_1 when it is reflected across the line

$y = x$. As the left half of Figure 7 makes clear, l_1 is reflected into a line l_2; moreover, their respective slopes m_1 and m_2 are related by $m_2 = 1/m_1$, provided $m_1 \neq 0$. If l_1 happens to be the tangent line to the graph of f at the point (c, d), then l_2 is the tangent line to the graph of f^{-1} at the point (d, c) (see the right half of Figure 7). We are led to the conclusion that

$$(f^{-1})'(d) = m_2 = \frac{1}{m_1} = \frac{1}{f'(c)}$$

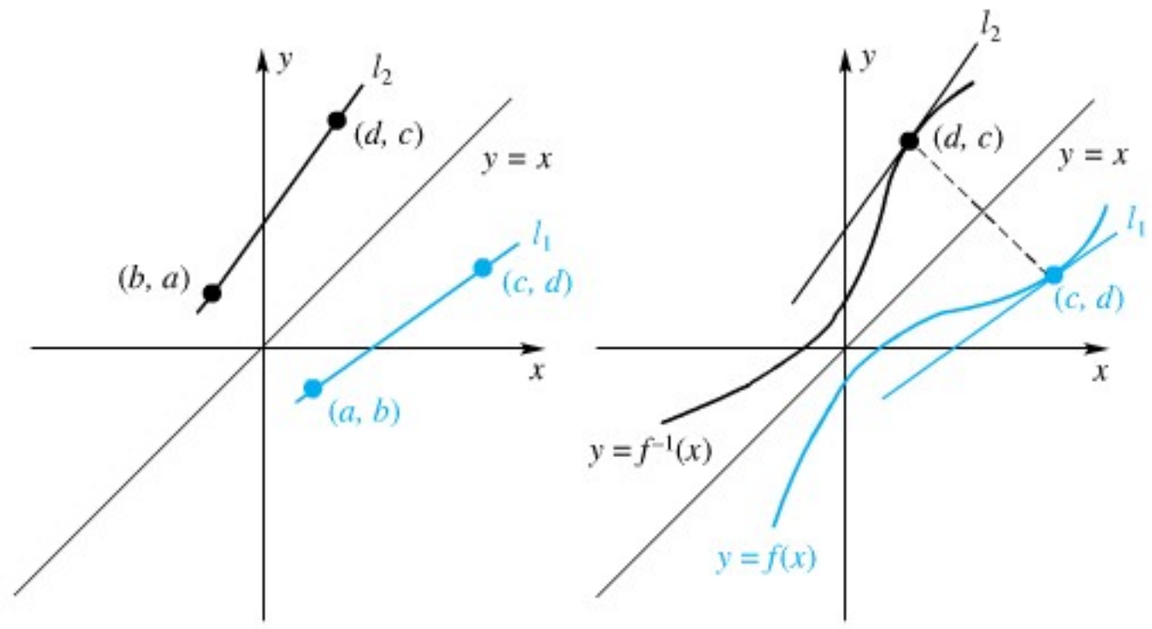


Figure 7

Pictures are sometimes deceptive, so we claim only to have made the following result plausible. For a formal proof, see any advanced calculus book.

Theorem B **Inverse Function Theorem**

Let f be differentiable and strictly monotonic on an interval I. If $f'(x) \neq 0$ at a certain x in I, then f^{-1} is differentiable at the corresponding point $y = f(x)$ in the range of f and

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

The conclusion to Theorem B is often written symbolically as

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

EXAMPLE 4 Let $y = f(x) = x^5 + 2x + 1$, as in Example 1. Find $(f^{-1})'(4)$.

SOLUTION Even though we cannot find a formula for f^{-1} in this case, we note that $y = 4$ corresponds to $x = 1$, and, since $f'(x) = 5x^4 + 2$,

$$(f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{5 + 2} = \frac{1}{7}$$ ■

Concepts Review

1. A f function is one-to-one if $x_1 \neq x_2$ implies _____.

2. A one-to-one function f has an inverse f^{-1} satisfying $f^{-1}(f(x)) =$ _____ and $f($_____$) = y$.

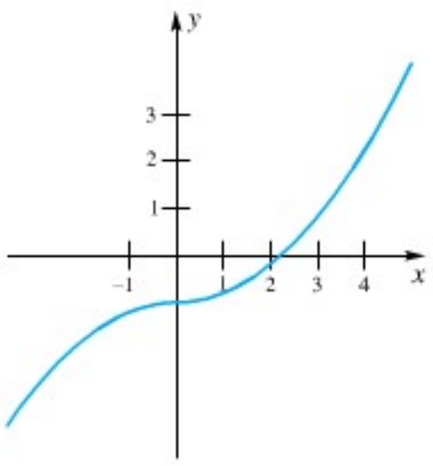
3. A useful criterion for f to be one-to-one (and so have an inverse) on a domain is that f be strictly _____ there. This means that f is either _____ or _____.

4. Let $y = f(x)$, where f has the inverse f^{-1}. The relation connecting the derivatives of f and f^{-1} is _____.

Problem Set 6.2

In Problems 1–6, the graph of $y = f(x)$ is shown. In each case, decide whether f has an inverse and, if so, estimate $f^{-1}(2)$.

1.



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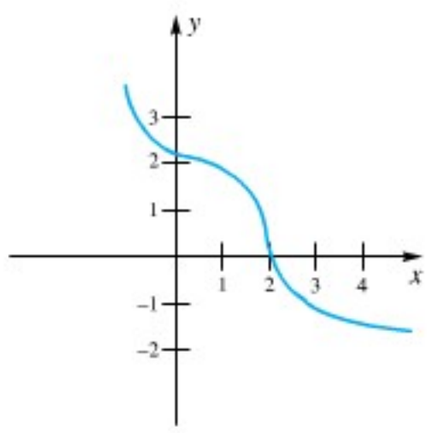

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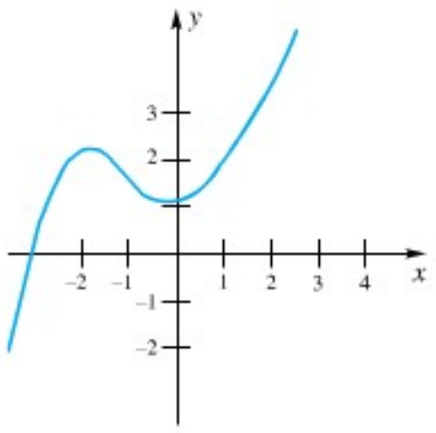

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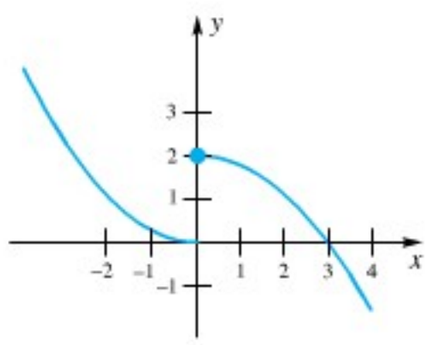

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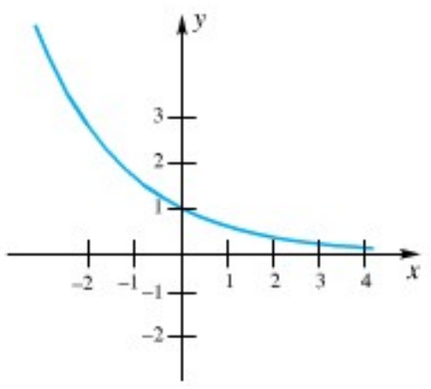

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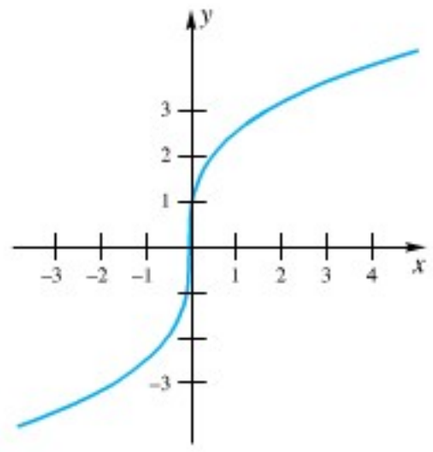


In Problems 7–14, show that f has an inverse by showing that it is strictly monotonic (see Example 1).

7. $f(x) = -x^5 - x^3$

8. $f(x) = x^7 + x^5$

9. $f(\theta) = \cos\theta, 0 \le \theta \le \pi$

10. $f(x) = \cot x = \dfrac{\cos x}{\sin x}, 0 < x < \dfrac{\pi}{2}$

11. $f(z) = (z-1)^2, z \ge 1$

12. $f(x) = x^2 + x - 6, x \ge 2$

13. $f(x) = \displaystyle\int_0^x \sqrt{t^4 + t^2 + 10}\, dt$

14. $f(r) = \displaystyle\int_r^1 \cos^4 t\, dt$

In Problems 15–28, find a formula for $f^{-1}(x)$ and then verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (see Examples 2 and 3).

15. $f(x) = x + 1$

16. $f(x) = -\dfrac{x}{3} + 1$

17. $f(x) = \sqrt{x+1}$

18. $f(x) = -\sqrt{1-x}$

19. $f(x) = -\dfrac{1}{x-3}$

20. $f(x) = \sqrt{\dfrac{1}{x-2}}$

21. $f(x) = 4x^2, x \le 0$

22. $f(x) = (x-3)^2, x \ge 3$

23. $f(x) = (x-1)^3$

24. $f(x) = x^{5/2}, x \ge 0$

25. $f(x) = \dfrac{x-1}{x+1}$

26. $f(x) = \left(\dfrac{x-1}{x+1}\right)^3$

27. $f(x) = \dfrac{x^3+2}{x^3+1}$

28. $f(x) = \left(\dfrac{x^3+2}{x^3+1}\right)^5$

29. Find the volume V of water in the conical tank of Figure 8 as a function of the height h. Then find the height h as a function of volume V.

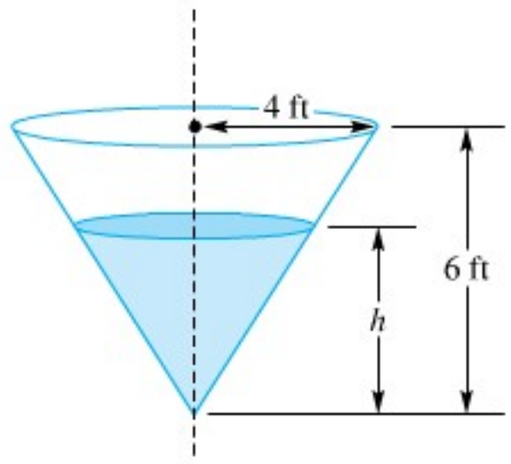


Figure 8

30. A ball is thrown vertically upward with velocity v_0. Find the maximum height H of the ball as a function of v_0. Then find the velocity v_0 required to achieve a height of H.

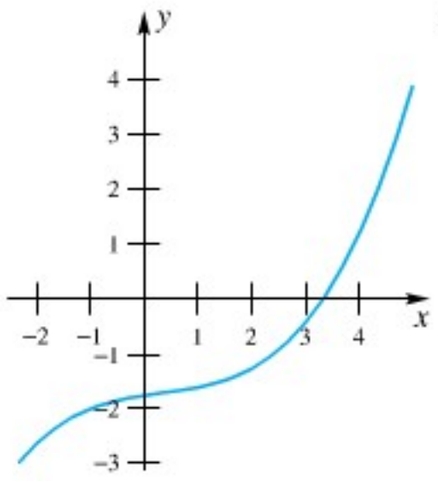
In Problems 31 and 32, restrict the domain of f so that f has an inverse, yet keeping its range as large as possible. Then find $f^{-1}(x)$. Suggestion: First graph f.

31. $f(x) = 2x^2 + x - 4$

32. $f(x) = x^2 - 3x + 1$

In each of Problems 33–36, the graph of $y = f(x)$ is shown. Sketch the graph of $y = f^{-1}(x)$ and estimate $(f^{-1})'(3)$.

33.



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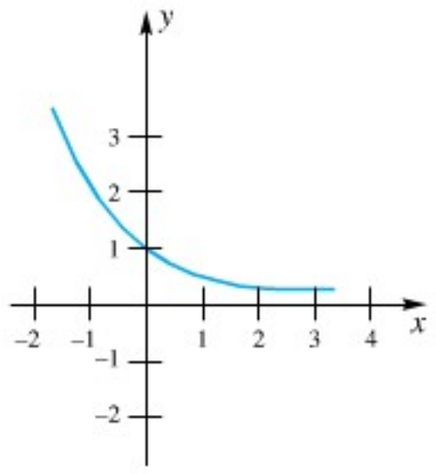

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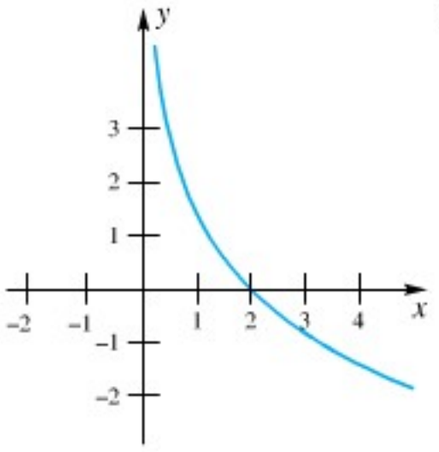

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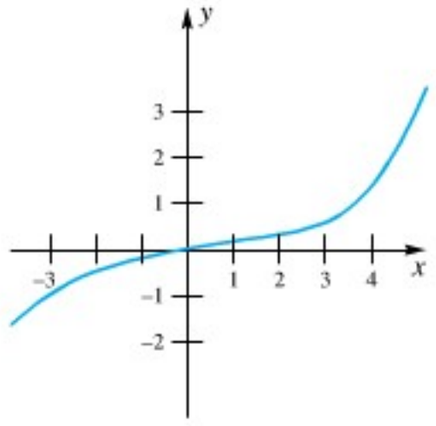


In Problems 37–40, find $(f^{-1})'(2)$ by using Theorem B (see Example 4). Note that you can find the x corresponding to $y = 2$ by inspection.

37. $f(x) = 3x^5 + x - 2$

38. $f(x) = x^5 + 5x - 4$

39. $f(x) = 2\tan x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

40. $f(x) = \sqrt{x+1}$

41. Suppose that both f and g have inverses and that $h(x) = (f \circ g)(x) = f(g(x))$. Show that h has an inverse given by $h^{-1} = g^{-1} \circ f^{-1}$.

42. Verify the result of Problem 41 for $f(x) = 1/x, g(x) = 3x + 2$.

43. If $f(x) = \displaystyle\int_0^x \sqrt{1+\cos^2 t}\, dt$, then f has an inverse. (Why?) Let $A = f(\pi/2)$ and $B = f(5\pi/6)$. Find

(a) $(f^{-1})'(A)$, (b) $(f^{-1})'(B)$,

(c) $(f^{-1})'(0)$.

44. Let $f(x) = \dfrac{ax+b}{cx+d}$ and assume $bc - ad \neq 0$.

(a) Find the formula for $f^{-1}(x)$.

(b) Why is the condition $bc - ad \neq 0$ needed?

(c) What condition on a, b, c, and d will make $f = f^{-1}$?

45. Suppose that f is continuous and strictly increasing on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. If $\displaystyle\int_0^1 f(x)\, dx = \frac{2}{5}$, calculate $\displaystyle\int_0^1 f^{-1}(y)\, dy$. *Hint:* Draw a picture.

EXPL **46.** Let f be continuous and strictly increasing on $[0, \infty)$ with $f(0) = 0$ and $f(x) \to \infty$ as $x \to \infty$. Use geometric reasoning to establish **Young's Inequality.** For $a > 0, b > 0$,

$$ab \leq \int_0^a f(x)\, dx + \int_0^b f^{-1}(y)\, dy$$

What is the condition for equality?

EXPL **47.** Let $p > 1, q > 1$, and $1/p + 1/q = 1$. Show that the inverse of $f(x) = x^{p-1}$ is $f^{-1}(y) = y^{q-1}$ and use this together with Problem 46 to prove **Minkowski's Inequality:**

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad a > 0, b > 0$$

Answers to Concepts Review: **1.** $f(x_1) \neq f(x_2)$ **2.** x; $f^{-1}(y)$ **3.** monotonic; increasing; decreasing **4.** $(f^{-1})'(y) = 1/f'(x)$

6.3 The Natural Exponential Function

The graph of $y = f(x) = \ln x$ was obtained at the end of Section 6.1 and is reproduced in Figure 1. The natural logarithm function is differentiable (hence continuous) and increasing on its domain $D = (0, \infty)$; its range is $R = (-\infty, \infty)$. It is, in fact, precisely the kind of function studied in Section 6.2, and therefore has an inverse $\ln^{-1}$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. This function is so important that it is given a special name and a special symbol.

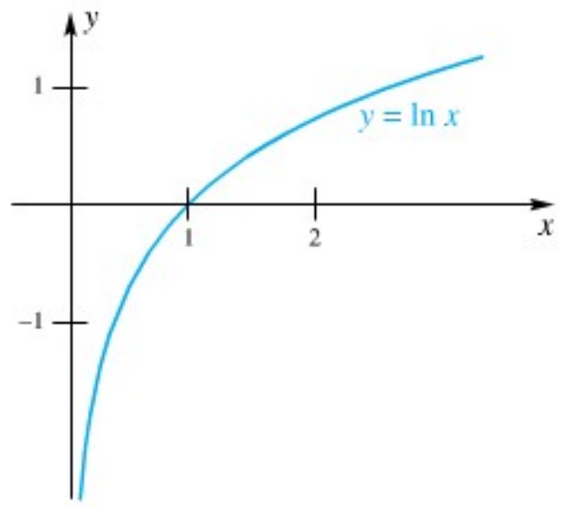


Figure 1

Definition

The inverse of ln is called the **natural exponential function** and is denoted by exp. Thus,

$$x = \exp y \iff y = \ln x$$

It follows immediately from this definition that

1. $\exp(\ln x) = x, \qquad x > 0$
2. $\ln(\exp y) = y, \qquad$ for all y

Since exp and ln are inverse functions, the graph of $y = \exp x$ is just the graph of $y = \ln x$ reflected across the line $y = x$ (Figure 2).

But why the name *exponential function*? You will see.

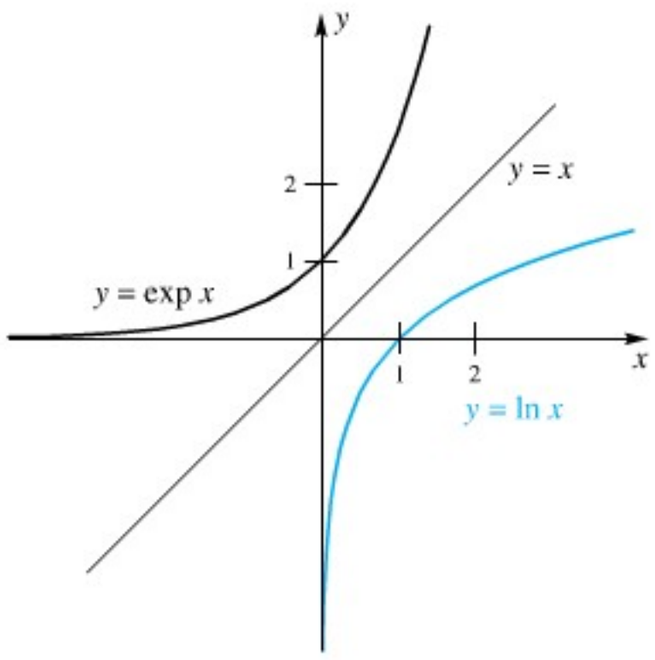


Figure 2

Properties of the Exponential Function

We begin by introducing a new number, which, like π, is so important in mathematics that it gets a special symbol, e. The letter e is appropriate since Leonhard Euler first recognized the significance of this number.

Definition

The letter e denotes the unique positive real number such that $\ln e = 1$.

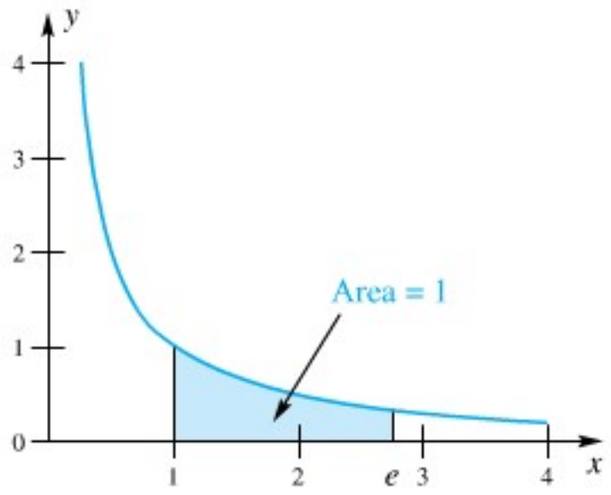


Figure 3

Figure 3 illustrates this definition; the area under the graph of $y = 1/x$ between $x = 1$ and $x = e$ is 1. Since $\ln e = 1$, it is also true that $\exp 1 = e$. The number e, like π, is irrational. Its decimal expansion is known to thousands of places; the first few digits are

$$e \approx 2.718281828459045$$

Now we make a crucial observation, one that depends only on facts already demonstrated: (1) above and Theorem 6.1A. If r is any rational number,

$$e^r = \exp(\ln e^r) = \exp(r \ln e) = \exp r$$

Let us emphasize the result. For rational r, $\exp r$ *is identical with* e^r. What was introduced in the most abstract way as the inverse of the natural logarithm, which itself was defined by an integral, has turned out to be a simple power.

But what if r is irrational? Here we remind you of a gap in all elementary algebra books. Never are irrational powers defined in anything approaching a rigorous manner. What is meant by $e^{\sqrt{2}}$? You will have a hard time pinning that number down, based on elementary algebra. But we must pin it down if we are to talk of such things as $D_x e^x$. Guided by what we learned above, we simply define e^x for all x (rational or irrational) by

$$e^x = \exp x$$

Note that (1) and (2) at the beginning of this section now take the following form:

$$(1)' \quad e^{\ln x} = x, \qquad x > 0$$
$$(2)' \quad \ln(e^y) = y, \quad \text{for all } y$$

Note also that $(1)'$ says that $\ln x$ is the *exponent* you need to put on e to get x. This is just the usual definition of the logarithm to the base e, as given in most precalculus books.

We can now easily prove two of the familiar laws of exponents.

Theorem A

Let a and b be any real numbers. Then $e^a e^b = e^{a+b}$ and $e^a/e^b = e^{a-b}$.

Proof To prove the first, we write

$$\begin{aligned} e^a e^b &= \exp(\ln e^a e^b) && \text{(by (1))} \\ &= \exp(\ln e^a + \ln e^b) && \text{(Theorem 6.1A)} \\ &= \exp(a + b) && \text{(by (2)')} \\ &= e^{a+b} && \text{(since } \exp x = e^x\text{)} \end{aligned}$$

The second fact is proved similarly. ■

Definitions of e

Authors choose different ways to define e.

1. $e = \ln^{-1} 1$ (our definition)
2. $e = \lim_{h \to 0} (1 + h)^{1/h}$
3. $e = \lim_{n \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$

In our book, definitions 2 and 3 become theorems (see Section 6.5, Theorem A, and Section 9.7, Example 3).

Phoenix

The number e appears throughout mathematics, but its importance rests most securely on its use as the base for the natural exponential function. And what makes this function so significant?

"Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising again from its own ashes, is its own derivative?"

François Le Lionnais

The Derivative of e^x Since exp and ln are inverses, we know from Theorem 6.2B that $\exp x = e^x$ is differentiable. To find a formula for $D_x e^x$, we could use that theorem. Alternatively, let $y = e^x$ so that

$$x = \ln y$$

Now differentiate both sides with respect to x. Using the Chain Rule, we obtain

$$1 = \frac{1}{y} D_x y$$

Thus,

$$D_x y = y = e^x$$

We have proved the remarkable fact that e^x is its own derivative; that is,

$$D_x e^x = e^x$$

Thus, $y = e^x$ is a solution of the differential equation $y' = y$.

If $u = f(x)$ is differentiable, then the Chain Rule yields

$$D_x e^u = e^u D_x u$$

EXAMPLE 1 Find $D_x e^{\sqrt{x}}$.

SOLUTION Using $u = \sqrt{x}$, we obtain

$$D_x e^{\sqrt{x}} = e^{\sqrt{x}} D_x \sqrt{x} = e^{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

■

EXAMPLE 2 Find $D_x e^{x^2 \ln x}$.

SOLUTION

$$\begin{aligned} D_x e^{x^2 \ln x} &= e^{x^2 \ln x} D_x (x^2 \ln x) \\ &= e^{x^2 \ln x} \left(x^2 \cdot \frac{1}{x} + 2x \ln x \right) \\ &= x e^{x^2 \ln x} (1 + \ln x^2) \end{aligned}$$

■

EXAMPLE 3 Let $f(x) = xe^{x/2}$. Find where f is increasing and decreasing, and where it is concave upward and downward. Also, identify all extreme values and points of inflection. Then, sketch the graph of f.

SOLUTION

$$f'(x) = \frac{xe^{x/2}}{2} + e^{x/2} = e^{x/2}\left(\frac{x+2}{2}\right)$$

and

$$f''(x) = \frac{e^{x/2}}{2} + \left(\frac{x+2}{2}\right)\frac{e^{x/2}}{2} = e^{x/2}\left(\frac{x+4}{4}\right)$$

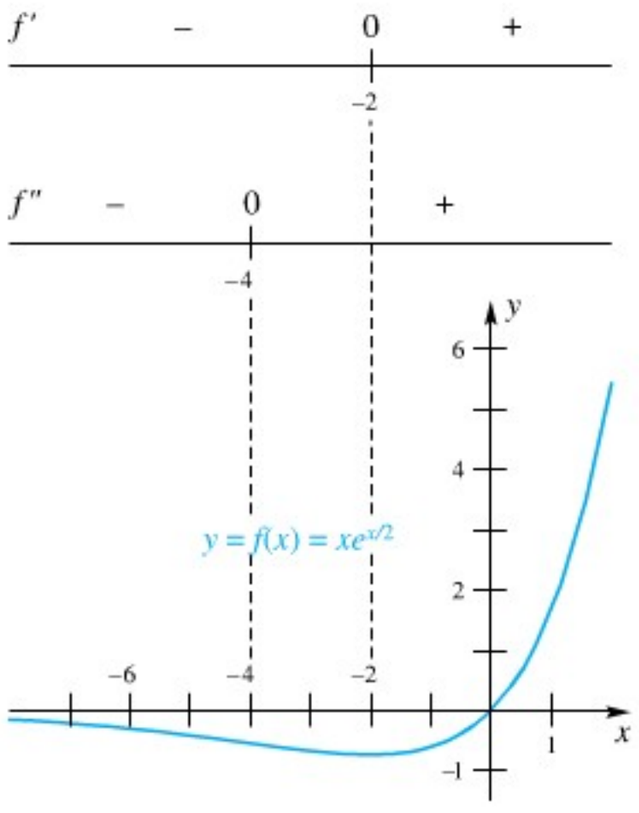


Figure 4

Keeping in mind that $e^{x/2} > 0$ for all x, we see that $f'(x) < 0$ for $x < -2$, $f'(-2) = 0$, and $f'(x) > 0$ for all $x > -2$. Thus, f is decreasing on $(-\infty, -2]$ and increasing on $[-2, \infty)$, and has its minimum value at $x = -2$ of $f(-2) = -2/e \approx -0.7$.

Also, $f''(x) < 0$ for $x < -4$, $f''(-4) = 0$, and $f''(x) > 0$ for $x > -4$; so the graph of f is concave downward on $(-\infty, -4)$ and concave upward on $(-4, \infty)$, and has a point of inflection at $(-4, -4e^{-2}) \approx (-4, -0.54)$. Since $\lim_{x \to -\infty} xe^{x/2} = 0$, the line $y = 0$ is a horizontal asymptote. This information supports the graph in Figure 4. ■

The derivative formula $D_x e^x = e^x$ automatically yields the integral formula $\int e^x \, dx = e^x + C$, or, with u replacing x,

$$\int e^u \, du = e^u + C$$

EXAMPLE 4 Evaluate $\int e^{-4x}\,dx$.

SOLUTION Let $u = -4x$, so $du = -4\,dx$. Then

$$\int e^{-4x}\,dx = -\frac{1}{4}\int e^{-4x}(-4\,dx) = -\frac{1}{4}\int e^{u}\,du = -\frac{1}{4}e^{u} + C = -\frac{1}{4}e^{-4x} + C$$ ■

EXAMPLE 5 Evaluate $\int x^2 e^{-x^3}\,dx$.

SOLUTION Let $u = -x^3$, so $du = -3x^2\,dx$. Then

$$\int x^2 e^{-x^3}\,dx = -\frac{1}{3}\int e^{-x^3}(-3x^2\,dx)$$
$$= -\frac{1}{3}\int e^{u}\,du = -\frac{1}{3}e^{u} + C$$
$$= -\frac{1}{3}e^{-x^3} + C$$ ■

EXAMPLE 6 Evaluate $\int_1^3 xe^{-3x^2}\,dx$.

SOLUTION Let $u = -3x^2$, so $du = -6x\,dx$. Then

$$\int xe^{-3x^2}\,dx = -\frac{1}{6}\int e^{-3x^2}(-6x\,dx) = -\frac{1}{6}\int e^{u}\,du$$
$$= -\frac{1}{6}e^{u} + C = -\frac{1}{6}e^{-3x^2} + C$$

Thus, by the Second Fundamental Theorem of Calculus,

$$\int_1^3 xe^{-3x^2}\,dx = \left[-\frac{1}{6}e^{-3x^2}\right]_1^3 = -\frac{1}{6}(e^{-27} - e^{-3}) = \frac{e^{-3} - e^{-27}}{6} \approx 0.0082978$$

The last result can be obtained directly with a calculator. ■

EXAMPLE 7 Evaluate $\int \frac{6e^{1/x}}{x^2}\,dx$.

SOLUTION Think of $\int e^{u}\,du$. Let $u = 1/x$, so $du = (-1/x^2)\,dx$. Then

$$\int \frac{6e^{1/x}}{x^2}\,dx = -6\int e^{1/x}\left(\frac{-1}{x^2}\,dx\right) = -6\int e^{u}\,du$$
$$= -6e^{u} + C = -6e^{1/x} + C$$ ■

Although the symbol e^y will largely supplant exp y throughout the rest of this book, exp occurs frequently in scientific writing, especially when the exponent y is complicated. For example, in statistics, one often encounters the normal probability density function, which is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Concepts Review

1. The function ln is _____ on $(0, \infty)$ and so has an inverse denoted by $\ln^{-1}$ or by _____.

2. The number e is defined in terms of ln by _____; its value to two decimal places is _____.

3. Since $e^x = \exp x = \ln^{-1} x$, it follows that $e^{\ln x} =$ _____ for $x > 0$, and $\ln(e^x) =$ _____.

4. Two remarkable facts about e^x are that $D_x(e^x) =$ _____ and $\int e^x\,dx =$ _____.

Problem Set 6.3

C **1.** Use your calculator to calculate each of the following.
Note: On some calculators there is an e^x button. On others you must press the INV (or 2nd) and ln x buttons.

(a) e^3 (b) $e^{2.1}$

(c) $e^{\sqrt{2}}$ (d) $e^{\cos(\ln 4)}$

C **2.** Calculate the following and explain why your answers are not surprising.

(a) $e^{3 \ln 2}$ (b) $e^{(\ln 64)/2}$

In Problems 3–10, simplify the given expression.

3. $e^{3 \ln x}$ **4.** $e^{-2 \ln x}$

5. $\ln e^{\cos x}$ **6.** $\ln e^{-2x-3}$

7. $\ln(x^3 e^{-3x})$ **8.** $e^{x-\ln x}$

9. $e^{\ln 3+2 \ln x}$ **10.** $e^{\ln x^2 - y \ln x}$

In Problems 11–22, find $D_x y$ (see Examples 1 and 2).

11. $y = e^{x+2}$ **12.** $y = e^{2x^2-x}$

13. $y = e^{\sqrt{x+2}}$ **14.** $y = e^{-1/x^2}$

15. $y = e^{2 \ln x}$ **16.** $y = e^{x/\ln x}$

17. $y = x^3 e^x$ **18.** $y = e^{x^3 \ln x}$

19. $y = \sqrt{e^{x^2}} + e^{\sqrt{x^2}}$ **20.** $y = e^{1/x^2} + 1/e^{x^2}$

21. $e^{xy} + xy = 2$ *Hint:* Use implicit differentiation.

22. $e^{x+y} = 4 + x + y$

23. Use your knowledge of the graph of $y = e^x$ to sketch the graphs of (a) $y = -e^x$ and (b) $y = e^{-x}$.

24. Explain why $a < b \Rightarrow e^{-a} > e^{-b}$.

In Problems 25–36, first find the domain of the given function f and then find where it is increasing and decreasing, and also where it is concave upward and downward. Identify all extreme values and points of inflection. Then sketch the graph of $y = f(x)$.

25. $f(x) = e^{2x}$ **26.** $f(x) = e^{-x/2}$

27. $f(x) = xe^{-x}$ **28.** $f(x) = e^x + x$

29. $f(x) = \ln(x^2 + 1)$ **30.** $f(x) = \ln(2x - 1)$

31. $f(x) = \ln(1 + e^x)$ **32.** $f(x) = e^{1-x^2}$

33. $f(x) = e^{-(x-2)^2}$ **34.** $f(x) = e^x - e^{-x}$

35. $f(x) = \int_0^x e^{-t^2}\,dt$ **36.** $f(x) = \int_0^x te^{-t}\,dt$

In Problems 37–44, find each integral.

37. $\int e^{3x+1}\,dx$ **38.** $\int xe^{x^2-3}\,dx$

39. $\int (x + 3)e^{x^2+6x}\,dx$ **40.** $\int \dfrac{e^x}{e^x - 1}\,dx$

41. $\int \dfrac{e^{-1/x}}{x^2}\,dx$ **42.** $\int e^{x+e^x}\,dx$

43. $\int_0^1 e^{2x+3}\,dx$ **44.** $\int_1^2 \dfrac{e^{3/x}}{x^2}\,dx$

45. Find the volume of the solid generated by revolving the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = \ln 3$ about the x-axis.

46. The region bounded by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$ is revolved about the y-axis. Find the volume of the resulting solid.

47. Find the area of the region bounded by the curve $y = e^{-x}$ and the line through the points $(0, 1)$ and $(1, 1/e)$.

48. Show that $f(x) = \dfrac{x}{e^x - 1} - \ln(1 - e^{-x})$ is decreasing for $x > 0$.

C **49. Stirling's Formula** says that for large n we can approximate $n! = 1 \cdot 2 \cdot 3 \cdots n$ by

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

(a) Calculate 10! exactly and then approximately using the above formula.

(b) Approximate 60!.

C **50.** It will be shown later (Section 9.9) that for small x

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Use this result to approximate $e^{0.3}$ and compare your result with what you get by calculating it directly. (Computers and calculators use sums like this to approximate e^x.)

51. Find the length of the curve given parametrically by $x = e^t \sin t$, $y = e^t \cos t$, $0 \le t \le \pi$.

C **52.** If customers arrive at a check-out counter at the average rate of k per minute, then (see books on probability theory) the probability that exactly n customers will arrive in a period of x minutes is given by the formula

$$P_n(x) = \frac{(kx)^n e^{-kx}}{n!},\ n = 0, 1, 2, \ldots$$

Find the probability that exactly 8 customers will arrive during a 30-minute period if the average arrival rate for this check-out counter is 1 customer every 4 minutes.

53. Let $f(x) = \dfrac{\ln x}{1 + (\ln x)^2}$ for x in $(0, \infty)$. Find

(a) $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to \infty} f(x)$;

(b) the maximum and minimum values of $f(x)$;

(c) $F'(\sqrt{e})$ if $F(x) = \int_1^{x^2} f(t)\,dt$.

54. Let R be the region bounded by $x = 0$, $y = e^x$, and the tangent line to $y = e^x$ that goes through the origin. Find

(a) the area of R;

(b) the volume of the solid obtained when R is revolved about the x-axis.

GC *Use a graphing calculator or a CAS to do Problems 55–60.*

55. Evaluate.

(a) $\int_{-3}^{3} \exp(-1/x^2)\,dx$ (b) $\int_0^{8\pi} e^{-0.1x} \sin x\,dx$

56. Evaluate.

(a) $\lim\limits_{x \to 0}(1 + x)^{1/x}$ (b) $\lim\limits_{x \to 0}(1 + x)^{-1/x}$

57. Find the area of the region between the graphs of $y = f(x) = \exp(-x^2)$ and $y = f''(x)$ on $[-3, 3]$.

EXPL **58.** Draw the graphs of $y = x^p e^{-x}$ for various positive values of p using the same axes. Make conjectures about

(a) $\lim_{x \to \infty} x^p e^{-x}$,

(b) the x-coordinate of the maximum point for $f(x) = x^p e^{-x}$.

59. Describe the behavior of $\ln(x^2 + e^{-x})$ for large negative x. For large positive x.

60. Draw the graphs of f and f', where $f(x) = 1/(1 + e^{1/x})$. Then determine each of the following:

(a) $\lim_{x \to 0^+} f(x)$ (b) $\lim_{x \to 0^-} f(x)$

(c) $\lim_{x \to \pm\infty} f(x)$ (d) $\lim_{x \to 0} f'(x)$

(e) The maximum and minimum values of f (if they exist).

Answers to Concepts Review: **1.** increasing; exp **2.** $\ln e = 1$; 2.72 **3.** x; x **4.** e^x; $e^x + C$

6.4 General Exponential and Logarithmic Functions

We defined $e^{\sqrt{2}}$, e^{π}, and all other irrational powers of e in the previous section. But what about $2^{\sqrt{2}}$, π^{π}, π^e, and similar irrational powers of other numbers? In fact, we want to give meaning to a^x for $a > 0$ and x any real number. Now, if $r = p/q$ is a rational number, then $a^r = \left(\sqrt[q]{a}\right)^p$. But we also know that

$$a^r = \exp(\ln a^r) = \exp(r \ln a) = e^{r \ln a}$$

This suggests the definition of the **exponential function to the base *a*.**

Definition

For $a > 0$ and any real number x,

$$a^x = e^{x \ln a}$$

Of course, this definition will be appropriate only if the usual properties of exponents are valid for it, a matter we take up shortly. To shore up our confidence in the definition, we use it to calculate 3^2 (with a little help from our calculator):

$$3^2 = e^{2 \ln 3} \approx e^{2(1.0986123)} \approx 9.000000$$

Your calculator may give a result that differs slightly from 9. Calculators use approximations for e^x and $\ln x$, and they round to a fixed number of decimal places (usually about 8).

Now we can fill a small gap in the properties of the natural logarithm left over from Section 6.1.

$$\ln(a^x) = \ln(e^{x \ln a}) = x \ln a$$

Thus, Property (iv) of Theorem 6.1A holds for all real x, not just rational x as claimed there. We will need this fact in the proof of Theorem A below.

Properties of a^x Theorem A summarizes the familiar properties of exponents, which can all be proved now in a completely rigorous manner. Theorem B shows us how to differentiate and integrate a^x.

Theorem A Properties of Exponents

If $a > 0$, $b > 0$, and x and y are real numbers, then

(i) $a^x a^y = a^{x+y}$; (ii) $\dfrac{a^x}{a^y} = a^{x-y}$;

(iii) $(a^x)^y = a^{xy}$; (iv) $(ab)^x = a^x b^x$;

(v) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$.

What is 2^{π}?

In algebra, 2^n is first defined for positive integers n. Thus, $2^1 = 2$ and $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$. Next, we define 2^n for zero,

$$2^0 = 1$$

and for negative integers:

$$2^{-n} = 1/2^n \quad \text{if} \quad n > 0$$

This means that $2^{-3} = 1/2^3 = 1/8$. Finally, we used root functions to define 2^r for rational numbers r. Thus,

$$2^{7/3} = \sqrt[3]{2^7}$$

Calculus is required to extend the definition of 2^x to the set of real numbers. One way to define 2^{π} would be to say that it is the limit of the sequence

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, \ldots$$

The definition we use is

$$2^{\pi} = e^{\pi \ln 2}$$

This definition involves calculus, because our definition of the natural log function involved the definite integral.

Proof We will prove (ii) and (iii), leaving the others for you.

(ii) $$\frac{a^x}{a^y} = e^{\ln(a^x/a^y)} = e^{\ln a^x - \ln a^y}$$
$$= e^{x \ln a - y \ln a} = e^{(x-y)\ln a} = a^{x-y}$$

(iii) $(a^x)^y = e^{y \ln a^x} = e^{yx \ln a} = a^{yx} = a^{xy}$ ■

Theorem B **Exponential Function Rules**

$$D_x a^x = a^x \ln a$$
$$\int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C, \qquad a \neq 1$$

Proof

$$D_x a^x = D_x(e^{x \ln a}) = e^{x \ln a} D_x(x \ln a)$$
$$= a^x \ln a$$

The integral formula follows immediately from the derivative formula. ■

EXAMPLE 1 Find $D_x\left(3^{\sqrt{x}}\right)$.

SOLUTION We use the Chain Rule with $u = \sqrt{x}$.

$$D_x\left(3^{\sqrt{x}}\right) = 3^{\sqrt{x}} \ln 3 \cdot D_x \sqrt{x} = \frac{3^{\sqrt{x}} \ln 3}{2\sqrt{x}}$$ ■

EXAMPLE 2 Find dy/dx if $y = (x^4 + 2)^5 + 5^{x^4+2}$.

SOLUTION

$$\frac{dy}{dx} = 5(x^4 + 2)^4 \cdot 4x^3 + 5^{x^4+2} \ln 5 \cdot 4x^3$$
$$= 4x^3[5(x^4 + 2)^4 + 5^{x^4+2} \ln 5]$$
$$= 20x^3[(x^4 + 2)^4 + 5^{x^4+1} \ln 5]$$ ■

EXAMPLE 3 Find $\int 2^{x^3} x^2 \, dx$.

SOLUTION Let $u = x^3$, so $du = 3x^2 \, dx$. Then

$$\int 2^{x^3} x^2 \, dx = \frac{1}{3} \int 2^{x^3} (3x^2 \, dx) = \frac{1}{3} \int 2^u \, du$$
$$= \frac{1}{3} \frac{2^u}{\ln 2} + C = \frac{2^{x^3}}{3 \ln 2} + C$$ ■

Why Other Bases?

Are bases other than e really needed? No. The formulas

$$a^x = e^{x \ln a}$$

and

$$\log_a x = \frac{\ln x}{\ln a}$$

allow us to turn any problem involving exponential functions or logarithmic functions with base a to corresponding functions with base e. This supports our terminology: *natural* exponential and *natural* logarithmic functions. It also explains the universal use of the latter functions in advanced work.

The Function $\log_a$ Finally, we are ready to make a connection with the logarithms that you studied in algebra. We note that if $0 < a < 1$ then $f(x) = a^x$ is a decreasing function; if $a > 1$, it is an increasing function, as you may check by considering the derivative. In either case, f has an inverse. We call this inverse the **logarithmic function to the base *a*.** This is equivalent to the following definition.

Definition

Let a be a positive number different from 1. Then

$$y = \log_a x \iff x = a^y$$

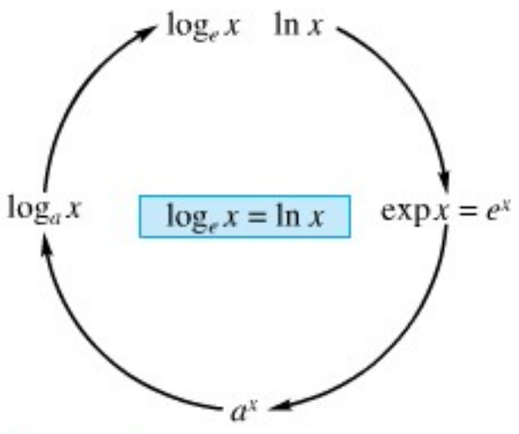


Figure 1

Historically, the most commonly used base was base 10, and the resulting logarithms were called **common logarithms.** But in calculus and all of advanced mathematics, the significant base is e. Notice that $\log_e$, being the inverse of $f(x) = e^x$, is just another symbol for ln; that is,

$$\log_e x = \ln x$$

We have come full circle (Figure 1). The function ln, which we introduced in Section 6.1, has turned out to be an ordinary logarithm, but to a rather special base, e.

Now observe that if $y = \log_a x$, so that $x = a^y$, then

$$\ln x = y \ln a$$

from which we conclude that

$$\log_a x = \frac{\ln x}{\ln a}$$

From this, it follows that $\log_a$ satisfies the usual properties associated with logarithms (see Theorem 6.1A). Also,

$$D_x \log_a x = \frac{1}{x \ln a}$$

EXAMPLE 4 If $y = \log_{10}(x^4 + 13)$, find $\dfrac{dy}{dx}$.

SOLUTION Let $u = x^4 + 13$ and apply the Chain Rule.

$$\frac{dy}{dx} = \frac{1}{(x^4 + 13) \ln 10} \cdot 4x^3 = \frac{4x^3}{(x^4 + 13) \ln 10}$$

■

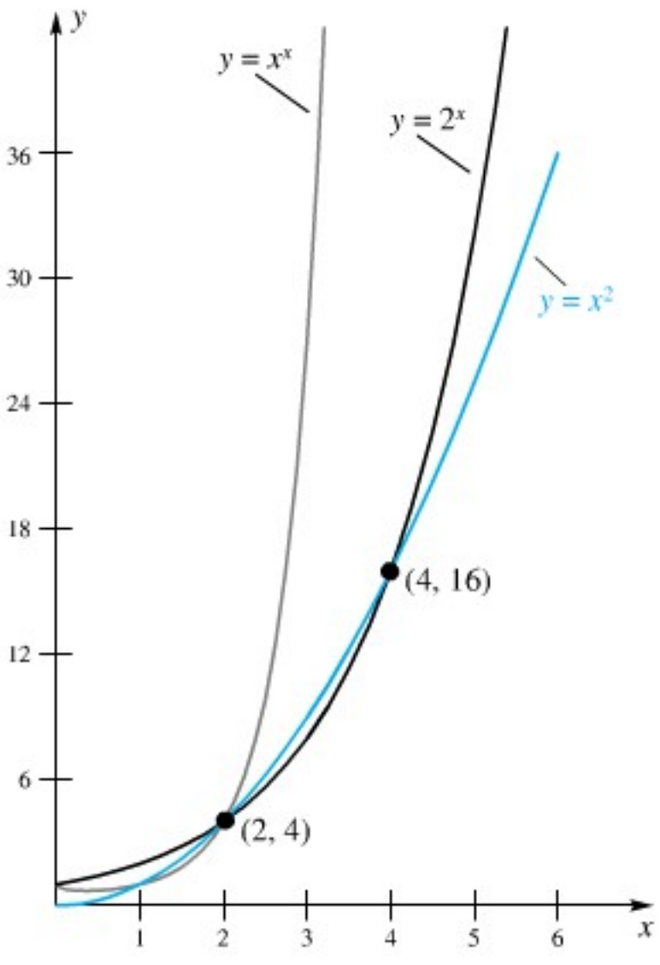


Figure 2

The Functions a^x, x^a, and x^x Begin by comparing the three graphs in Figure 2. More generally, let a be a constant. Do not confuse $f(x) = a^x$, an *exponential function*, with $g(x) = x^a$, a *power function*. And do not confuse their derivatives. We have just learned that

$$D_x(a^x) = a^x \ln a$$

What about $D_x(x^a)$? For a rational, we proved the Power Rule in Chapter 2, which says that

$$D_x(x^a) = ax^{a-1}$$

Now we assert that this is true even if a is irrational. To see this, write

$$D_x(x^a) = D_x(e^{a \ln x}) = e^{a \ln x} \cdot \frac{a}{x}$$

$$= x^a \cdot \frac{a}{x} = ax^{a-1}$$

The corresponding rule for integrals also holds even if a is irrational.

$$\int x^a \, dx = \frac{x^{a+1}}{a + 1} + C, \qquad a \neq -1$$

Finally, we consider $f(x) = x^x$, a variable to a variable power. There is a formula for $D_x(x^x)$, but we do not recommend that you memorize it. Rather, we suggest that you learn two methods for finding it, as illustrated below.

EXAMPLE 5 If $y = x^x$, $x > 0$, find $D_x y$ by two different methods.

SOLUTION

Method 1 We may write

$$y = x^x = e^{x \ln x}$$

Thus, using the Chain Rule and the Product Rule,

$$D_x y = e^{x \ln x} D_x(x \ln x) = x^x\left(x \cdot \frac{1}{x} + \ln x\right) = x^x(1 + \ln x)$$

Method 2 Recall the *logarithmic differentiation* technique from Section 6.1.

$$y = x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} D_x y = x \cdot \frac{1}{x} + \ln x$$

$$D_x y = y(1 + \ln x) = x^x(1 + \ln x)$$ ■

EXAMPLE 6 If $y = (x^2 + 1)^\pi + \pi^{\sin x}$, find dy/dx.

SOLUTION

$$\frac{dy}{dx} = \pi(x^2 + 1)^{\pi - 1}(2x) + \pi^{\sin x} \ln \pi \cdot \cos x$$ ■

From a^x to $[f(x)]^{g(x)}$

Note the increasing complexity of the functions that we have considered. The progression a^x to x^a to x^x is one chain. A more complex chain is $a^{f(x)}$ to $[f(x)]^a$ to $[f(x)]^{g(x)}$. We now know how to find the derivatives of all these functions. Finding the derivative of the last of these is best accomplished by logarithmic differentiation, a technique introduced in Section 6.1 and illustrated in Examples 5 and 7.

EXAMPLE 7 If $y = (x^2 + 1)^{\sin x}$, find $\dfrac{dy}{dx}$.

SOLUTION We use logarithmic differentiation.

$$\ln y = (\sin x) \ln(x^2 + 1)$$

$$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{2x}{x^2 + 1} + (\cos x) \ln(x^2 + 1)$$

$$\frac{dy}{dx} = (x^2 + 1)^{\sin x}\left[\frac{2x \sin x}{x^2 + 1} + (\cos x) \ln(x^2 + 1)\right]$$ ■

EXAMPLE 8 Evaluate $\displaystyle\int_{1/2}^{1} \frac{5^{1/x}}{x^2}\, dx$.

SOLUTION Let $u = 1/x$, so $du = (-1/x^2)\, dx$. Then

$$\int \frac{5^{1/x}}{x^2}\, dx = -\int 5^{1/x}\left(-\frac{1}{x^2}\, dx\right) = -\int 5^u\, du$$

$$= -\frac{5^u}{\ln 5} + C = -\frac{5^{1/x}}{\ln 5} + C$$

Thus, by the Second Fundamental Theorem of Calculus,

$$\int_{1/2}^{1} \frac{5^{1/x}}{x^2} dx = \left[-\frac{5^{1/x}}{\ln 5}\right]_{1/2}^{1} = \frac{1}{\ln 5}(5^2 - 5)$$

$$= \frac{20}{\ln 5} \approx 12.43$$ ■

Concepts Review

1. In terms of e and ln, $\pi^{\sqrt{3}} =$ _____. More generally $a^x =$ _____.

2. $\ln x = \log_a x$, where $a =$ _____.

3. $\log_a x$ can be expressed in terms of ln by $\log_a x =$ _____.

4. The derivative of the power function $f(x) = x^a$ is $f'(x) =$ _____; the derivative of the exponential function $g(x) = a^x$ is $g'(x) =$ _____.

Problem Set 6.4

In Problems 1–8, solve for x. Hint: $\log_a b = c \Leftrightarrow a^c = b$.

1. $\log_2 8 = x$

2. $\log_5 x = 2$

3. $\log_4 x = \frac{3}{2}$

4. $\log_x 64 = 4$

5. $2\log_9\left(\frac{x}{3}\right) = 1$

6. $\log_4\left(\frac{1}{2x}\right) = 3$

7. $\log_2(x + 3) - \log_2 x = 2$

8. $\log_5(x + 3) - \log_5 x = 1$

C *Use* $\log_a x = (\ln x)/(\ln a)$ *to calculate each of the logarithms in Problems 9–12.*

9. $\log_5 12$

10. $\log_7(0.11)$

11. $\log_{11}(8.12)^{1/5}$

12. $\log_{10}(8.57)^7$

C *In Problems 13–16, use natural logarithms to solve each of the exponential equations. Hint: To solve* $3^x = 11$, *take ln of both sides, obtaining* $x \ln 3 = \ln 11$; *then* $x = (\ln 11)/(\ln 3) \approx 2.1827$.

13. $2^x = 17$

14. $5^x = 13$

15. $5^{2s-3} = 4$

16. $12^{1/(\theta-1)} = 4$

In Problems 17–26, find the indicated derivative or integral.

17. $D_x(6^{2x})$

18. $D_x(3^{2x^2-3x})$

19. $D_x \log_3 e^x$

20. $D_x \log_{10}(x^3 + 9)$

21. $D_z[3^z \ln(z + 5)]$

22. $D_\theta\sqrt{\log_{10}(3^{\theta^2-\theta})}$

23. $\int x2^{x^2}\,dx$

24. $\int 10^{5x-1}\,dx$

25. $\int_1^4 \frac{5^{\sqrt{x}}}{\sqrt{x}}\,dx$

26. $\int_0^1 (10^{3x} + 10^{-3x})\,dx$

In Problems 27–32, find dy/dx. *Note: You must distinguish among problems of the type* a^x, x^a, *and* x^x *as in Examples 5–7.*

27. $y = 10^{(x^2)} + (x^2)^{10}$

28. $y = \sin^2 x + 2^{\sin x}$

29. $y = x^{\pi+1} + (\pi + 1)^x$

30. $y = 2^{(e^x)} + (2^e)^x$

31. $y = (x^2 + 1)^{\ln x}$

32. $y = (\ln x^2)^{2x+3}$

33. If $f(x) = x^{\sin x}$, find $f'(1)$.

C **34.** Let $f(x) = \pi^x$ and $g(x) = x^\pi$. Which is larger, $f(e)$ or $g(e)$? $f'(e)$ or $g'(e)$?

In Problems 35–40, first find the domain of the given function f and then find where it is increasing and decreasing, and also where it is concave upward and downward. Identify all extreme values and points of inflection. Then sketch the graph of $y = f(x)$.

35. $f(x) = 2^{-x}$

36. $f(x) = x2^{-x}$

37. $f(x) = \log_2(x^2 + 1)$

38. $f(x) = x\log_3(x^2 + 1)$

39. $f(x) = \int_1^x 2^{-t^2}\,dt$

40. $f(x) = \int_0^x \log_{10}(t^2 + 1)\,dt$

41. How are $\log_{1/2} x$ and $\log_2 x$ related?

42. Sketch the graphs of $\log_{1/3} x$ and $\log_3 x$ using the same coordinate axes.

C **43.** The magnitude M of an earthquake on the **Richter scale** is

$$M = 0.67\log_{10}(0.37E) + 1.46$$

where E is the energy of the earthquake in kilowatt-hours. Find the energy of an earthquake of magnitude 7. Of magnitude 8.

C **44.** The loudness of sound is measured in decibels in honor of Alexander Graham Bell (1847–1922), inventor of the telephone. If the variation in pressure is P pounds per square inch, then the loudness L in decibels is

$$L = 20\log_{10}(121.3P)$$

Find the variation in pressure caused by music at 115 decibels.

C **45.** In the equally tempered scale to which keyed instruments have been tuned since the days of J.S. Bach (1685–1750), the frequencies of successive notes C, C#, D, D#, E, F, F#, G, G#, A, A#, B, C form a geometric sequence (progression), with C having twice the frequency of C (C# is read C sharp). What is the ratio r between the frequencies of successive notes? If the frequency of A is 440, find the frequency of $\overline{C}$.

46. Prove that $\log_2 3$ is irrational. *Hint:* Use proof by contradiction.

GC **47.** You suspect that the xy-data that you collect lie on either an exponential curve $y = Ab^x$ or a power curve $y = Cx^d$. To check, you plot ln y against x on one graph and ln y against ln x on another graph. (Graphing calculators and CASs have options to make the vertical axis, or both the vertical and horizontal axes, a logarithmic scale.) Explain how these graphs will help you to come to a conclusion.

48. (An Amusement) Given the problem of finding y' if $y = x^x$, student A did the following:

Wrong 1

$$\begin{aligned} y &= x^x \\ y' &= x \cdot x^{x-1} \cdot 1 \\ &= x^x \end{aligned} \qquad \begin{pmatrix}\text{misapplying the}\\ \text{Power Rule}\end{pmatrix}$$

Student B did this:

Wrong 2

$$\begin{aligned} y &= x^x \\ y' &= x^x \cdot \ln x \cdot 1 \\ &= x^x \ln x \end{aligned} \qquad \begin{pmatrix}\text{misapplying the}\\ \text{Exponential}\\ \text{Function Rule}\end{pmatrix}$$

The sum $x^x + x^x \ln x$ is correct (Example 5), so

$$\text{WRONG 1} + \text{WRONG 2} = \text{RIGHT}$$

Show that the same procedure yields a correct answer for finding the derivative of $y = f(x)^{g(x)}$.

49. Convince yourself that $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$ are not the same function. Then find $f'(x)$ and $g'(x)$. *Note:* When mathematicians write x^{x^x}, they mean $x^{(x^x)}$.

50. Consider $f(x) = \dfrac{a^x - 1}{a^x + 1}$ for fixed a, $a > 0$, $a \neq 1$. Show that f has an inverse and find a formula for $f^{-1}(x)$.

51. For a fixed $a > 1$, let $f(x) = x^a/a^x$ on $[0, \infty)$. Show:

(a) $\lim_{x\to\infty} f(x) = 0$ *Hint:* Study $\ln f(x)$;

(b) $f(x)$ is maximized at $x_0 = a/\ln a$;

(c) $x^a = a^x$ has two positive solutions if $a \neq e$ and only one such solution if $a = e$;

(d) $\pi^e < e^\pi$.

52. Let $f_u(x) = x^u e^{-x}$ for $x \geq 0$. Show that for any fixed $u > 0$:

(a) $f_u(x)$ attains its maximum at $x_0 = u$;

(b) $f_u(u) > f_u(u + 1)$ and $f_{u+1}(u + 1) > f_{u+1}(u)$ imply

$$\left(\frac{u+1}{u}\right)^u < e < \left(\frac{u+1}{u}\right)^{u+1}$$

(c) $$\frac{u}{u+1}\, e < \left(\frac{u+1}{u}\right)^u < e.$$

Conclude from part (c) that $\lim_{u\to\infty}\left(1 + \dfrac{1}{u}\right)^u = e$.

GC **53.** Find $\lim_{x\to 0^+} x^x$. Also find the coordinates of the minimum point for $f(x) = x^x$ on $[0, 4]$.

GC **54.** Draw the graphs of $y = x^3$ and $y = 3^x$ using the same axes and find all their intersection points.

CAS **55.** Evaluate $\displaystyle\int_0^{4\pi} x^{\sin x}\, dx$.

CAS **56.** Refer to Problem 49. Draw the graphs of f and g using the same axes. Then draw the graphs of f' and g' using the same axes.

Our graphing experience so far has been restricted to using standard (linear) coordinate spacings. When working with exponential and logarithmic functions it may be more instructive to use logarithmic and log–log scales. We explore these techniques in Problems 57 and 58.

GC **57.** On a single set of axes, use your calculator to draw the graphs of $y = 2^x$, $y = 3^x$, and $y = 4^x$ over the interval $0 < x < 4$. Do the same for the inverse functions $y = \log_2 x$, $y = \log_3 x$, and $y = \log_4 x$. If we use a computer graphing program that permits the use of semilog axes (a logarithmic scale on the y-axis and a normal scale on the x-axis) to graph the functions $y = 2^x$, $y = 3^x$, and $y = 4^x$ over the region $-5 < x < 5$ (Figure 3), we get three lines.

(a) Identify each of the lines in Figure 3.

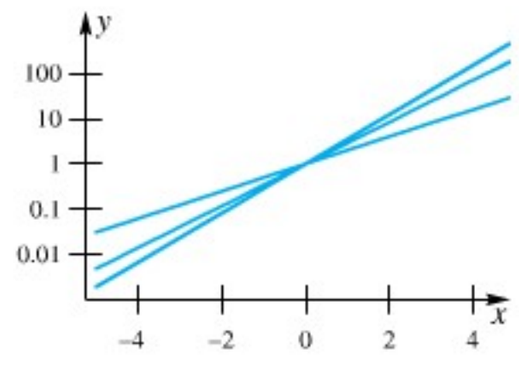


Figure 3

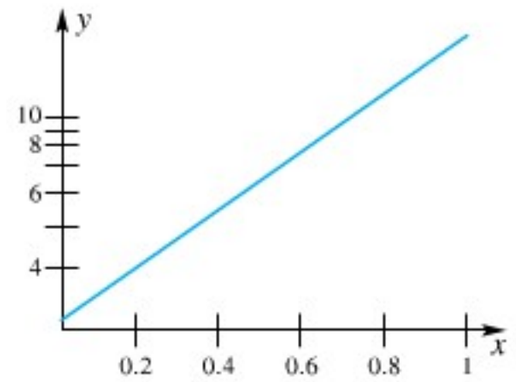


Figure 4

(b) Noting that, if $y = Cb^x$ then $\ln y = \ln C + x \ln b$, explain why all the curves in Figure 3 are lines through the point $(0, 1)$.

(c) Based on the semilog plot given by Figure 4, determine the C and b in the equation $y = Cb^x$.

58. If we use log scaling for the x-axis as well as the y-axis (called a log–log plot) and graph several power functions, we will also get lines. Using the result that, upon taking logs, $y = Cx^r$ becomes $\log y = \log C + r \log x$, identify the equations that are graphed in Figure 5.

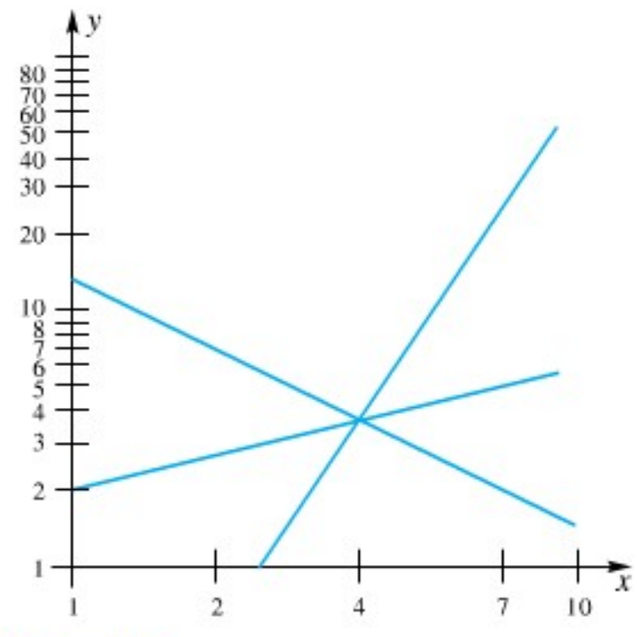


Figure 5

Answers to Concepts Review: **1.** $e^{\sqrt{3}\ln \pi}$; $e^{x \ln a}$ **2.** e **3.** $(\ln x)/(\ln a)$ **4.** ax^{a-1}; $a^x \ln a$

6.8 The Inverse Trigonometric Functions and Their Derivatives

The six basic trigonometric functions (sine, cosine, tangent, cotangent, secant, and cosecant) were defined in Section 0.7, and we have used them occasionally in examples and problems. With respect to the notion of inverse, they are miserable functions, since for each y in their range there are infinitely many x's that correspond to it (Figure 1). Nonetheless, we are going to introduce a notion of inverse for them. That this is possible rests on a procedure called **restricting the domain,** which was discussed briefly in Section 6.2.

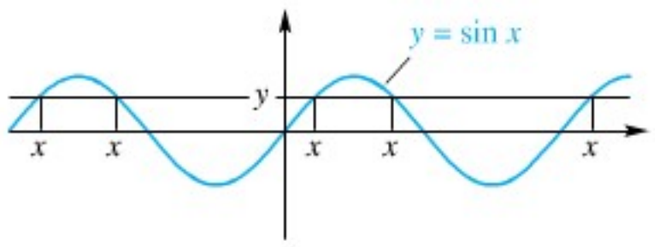


Figure 1

Inverse Sine and Inverse Cosine

In the case of sine and cosine, we restrict the domain, keeping the range as large as possible while insisting that the resulting function have an inverse. This can be done in many ways, but the agreed procedure is suggested by Figures 2 and 3. We also show the graph of the corresponding inverse function, obtained, as usual, by reflecting across the line $y = x$.

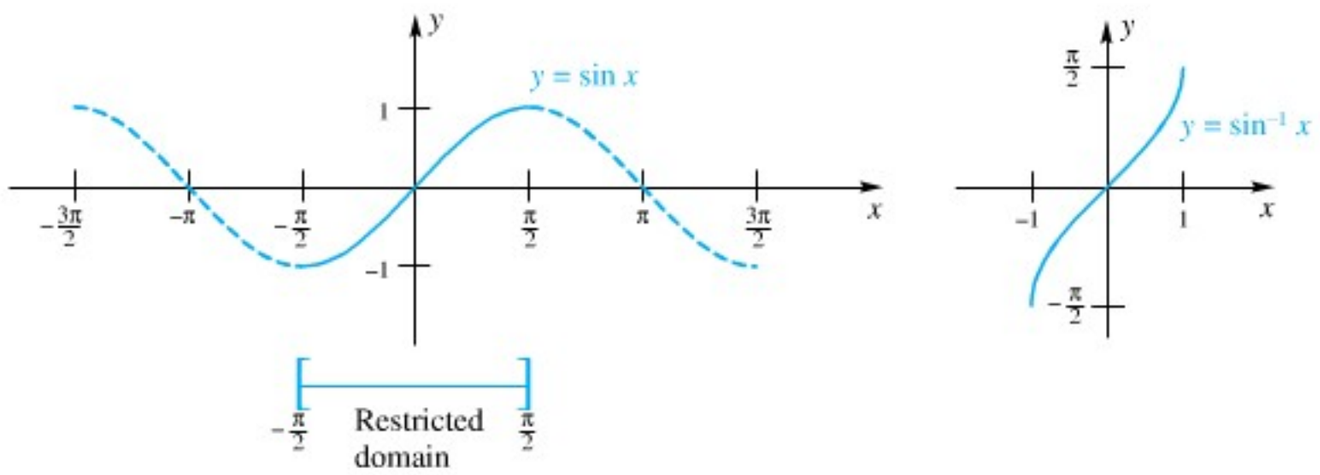


Figure 2

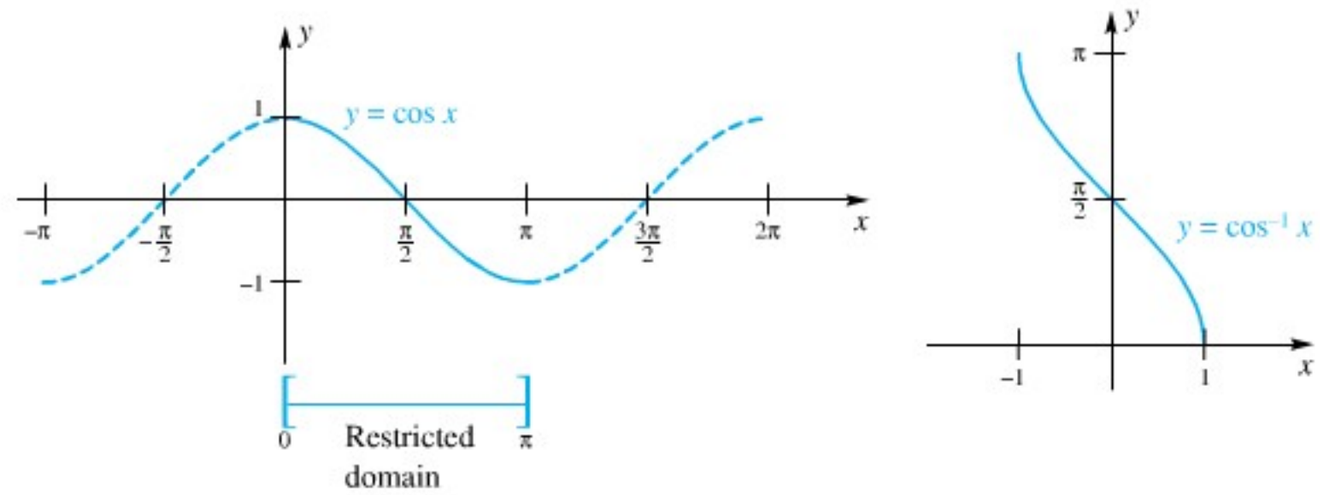


Figure 3

We formalize what we have shown in a definition.

Definition

To obtain inverses for sine and cosine, we restrict their domains to $[-\pi/2, \pi/2]$ and $[0, \pi]$, respectively. Thus,

$$x = \sin^{-1} y \quad \Leftrightarrow \quad y = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$x = \cos^{-1} y \quad \Leftrightarrow \quad y = \cos x, 0 \leq x \leq \pi$$

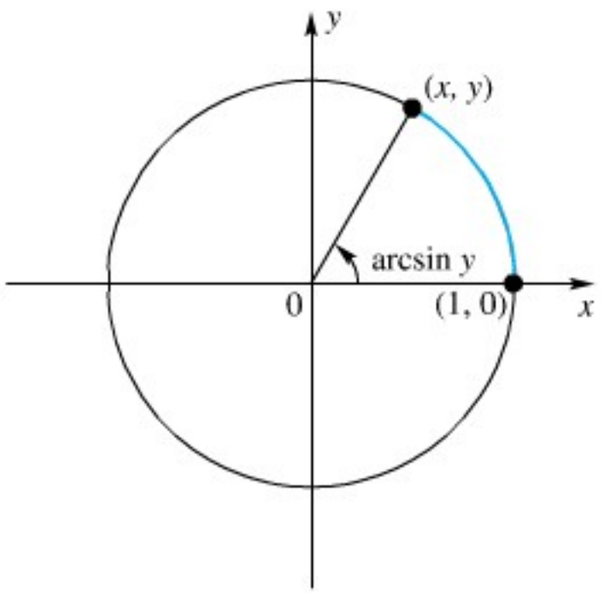


Figure 4

The symbol arcsin is often used for $\sin^{-1}$, and arccos is similarly used for $\cos^{-1}$. Think of arcsin as meaning "the arc whose sine is" or "the angle whose sine is" (Figure 4). We will use both forms throughout the rest of this book.

EXAMPLE 1 Calculate

(a) $\sin^{-1}\left(\sqrt{2}/2\right)$, (b) $\cos^{-1}\left(-\frac{1}{2}\right)$,

(c) $\cos(\cos^{-1} 0.6)$, and (d) $\sin^{-1}(\sin 3\pi/2)$

SOLUTION

(a) $\sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$ (b) $\cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$

(c) $\cos(\cos^{-1} 0.6) = 0.6$ (d) $\sin^{-1}\left(\sin\dfrac{3\pi}{2}\right) = -\dfrac{\pi}{2}$

The only one of these that is tricky is (d). Note that it would be wrong to give $3\pi/2$ as the answer, since $\sin^{-1} y$ is always in the interval $[-\pi/2, \pi/2]$. Work the problem in steps, as follows.

$$\sin^{-1}\left(\sin\frac{3\pi}{2}\right) = \sin^{-1}(-1) = -\pi/2$$ ■

EXAMPLE 2 Use a calculator to find

(a) $\cos^{-1}(-0.61)$, (b) $\sin^{-1}(1.21)$, (c) $\sin^{-1}(\sin 4.13)$

SOLUTION Use a calculator in radian mode. It has been programmed to give answers that are consistent with the definitions that we have given.

(a) $\cos^{-1}(-0.61) = 2.2268569$

(b) Your calculator should indicate an error, since $\sin^{-1}(1.21)$ does not exist.

(c) $\sin^{-1}(\sin 4.13) = -0.9884073$ ■

Another Way To Say It

$\sin^{-1} y$

is the number in the interval $[-\pi/2, \pi/2]$ whose sine is y.

$\cos^{-1} y$

is the number in the interval $[0, \pi]$ whose cosine is y.

$\tan^{-1} y$

is the number in the interval $(-\pi/2, \pi/2)$ whose tangent is y.

Inverse Tangent and Inverse Secant In Figure 5, we show the graph of the tangent function, its restricted domain, and the graph of $y = \tan^{-1} x$.

There is a standard way to restrict the domain of the cotangent function, that is, to $(0, \pi)$, so that it has an inverse. However, this function does not play a significant role in calculus.

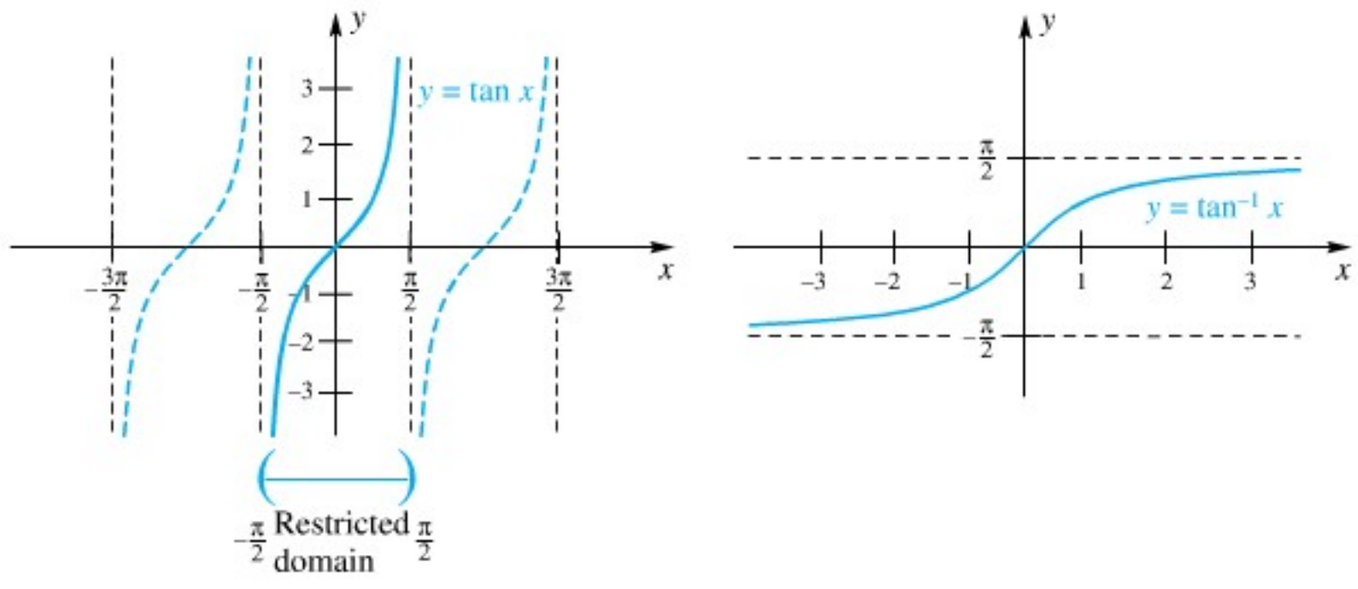


Figure 5

To obtain an inverse for secant, we graph $y = \sec x$, restrict its domain appropriately, and then graph $y = \sec^{-1} x$ (Figure 6).

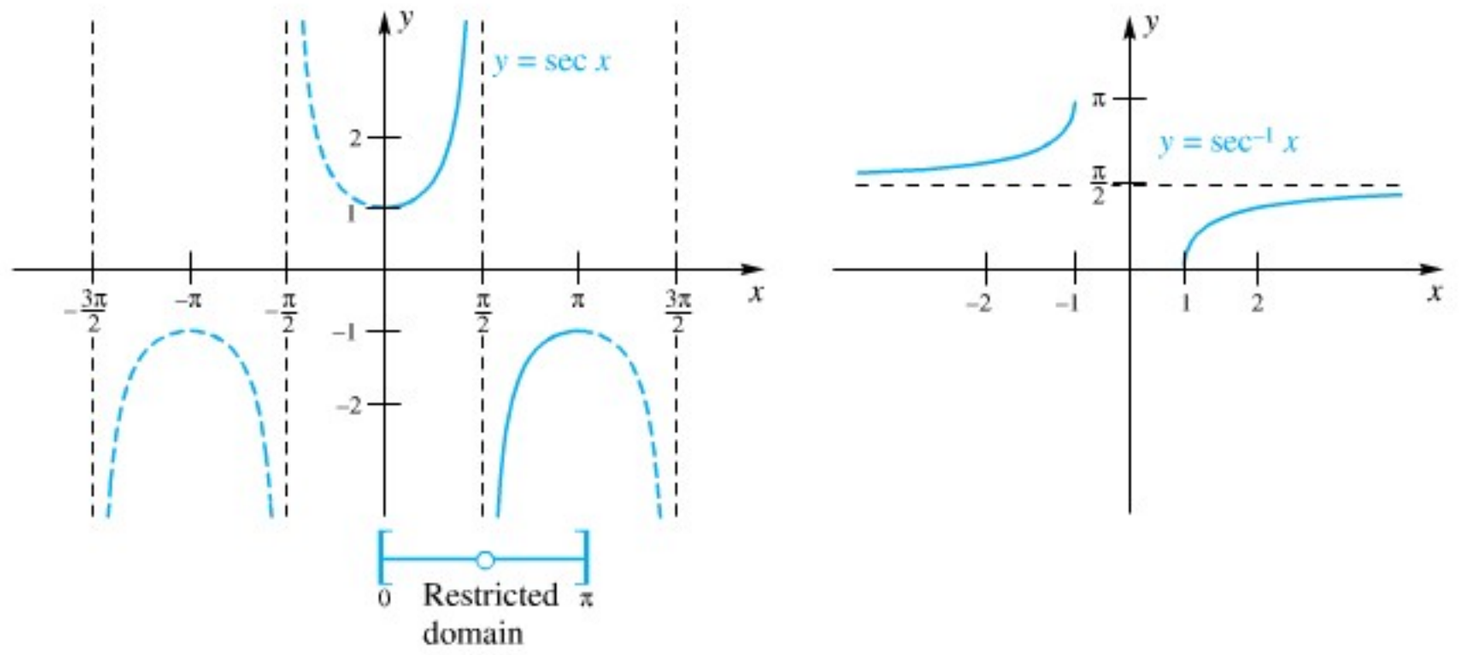


Figure 6

Definition

To obtain inverses for tangent and secant, we restrict their domains to $(-\pi/2, \pi/2)$ and $[0, \pi/2) \cup (\pi/2, \pi]$, respectively. Thus,

$$x = \tan^{-1} y \quad \Leftrightarrow \quad y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$x = \sec^{-1} y \quad \Leftrightarrow \quad y = \sec x, 0 \le x \le \pi, x \ne \frac{\pi}{2}$$

Some authors restrict the domain of the secant in a different way. Thus, if you refer to another book, you must check that author's definition. We will have no need to define $\csc^{-1}$, though this can also be done.

EXAMPLE 3 Calculate

(a) $\tan^{-1}(1)$,
(b) $\tan^{-1}\left(-\sqrt{3}\right)$,
(c) $\tan^{-1}(\tan 5.236)$,
(d) $\sec^{-1}(-1)$,
(e) $\sec^{-1}(2)$, and
(f) $\sec^{-1}(-1.32)$

SOLUTION

(a) $\tan^{-1}(1) = \dfrac{\pi}{4}$
(b) $\tan^{-1}\left(-\sqrt{3}\right) = -\dfrac{\pi}{3}$
(c) $\tan^{-1}(\tan 5.236) = -1.0471853$

Most of us have trouble remembering our secants; moreover, most calculators fail to have a secant button. Therefore, we suggest that you remember that $\sec x = 1/\cos x$. From this, it follows that

$$\sec^{-1} y = \cos^{-1}\left(\frac{1}{y}\right)$$

and this allows us to use known facts about the cosine.

(d) $\sec^{-1}(-1) = \cos^{-1}(-1) = \pi$

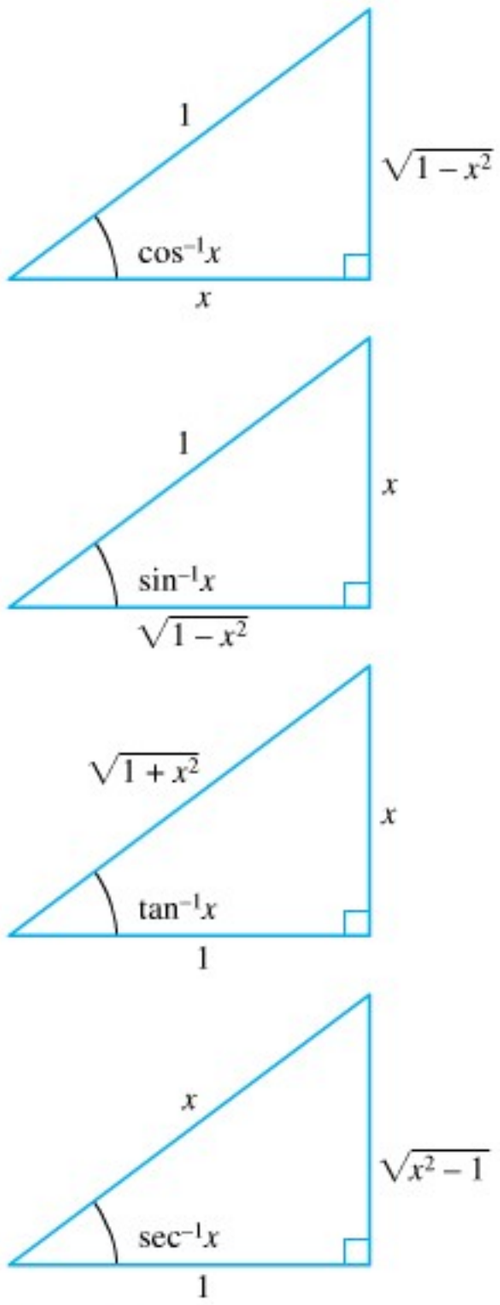


Figure 7

(e) $\sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

(f) $\sec^{-1}(-1.32) = \cos^{-1}\left(-\frac{1}{1.32}\right) = \cos^{-1}(0.7575758)$

$= 2.4303875$ ■

Four Useful Identities Theorem A gives some useful identities. You can recall them by reference to the triangles in Figure 7.

Theorem A

(i) $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$

(ii) $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

(iii) $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$

(iv) $\tan(\sec^{-1} x) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -\sqrt{x^2 - 1}, & \text{if } x \leq -1 \end{cases}$

Proof To prove (i), recall that $\sin^2\theta + \cos^2\theta = 1$. If $0 \leq \theta \leq \pi$, then

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

Now apply this with $\theta = \cos^{-1} x$ and use the fact that $\cos(\cos^{-1} x) = x$ to get

$$\sin(\cos^{-1} x) = \sqrt{1 - \cos^2(\cos^{-1} x)} = \sqrt{1 - x^2}$$

Identity (ii) is proved in a completely similar manner. To prove (iii) and (iv), use the identity $\sec^2\theta = 1 + \tan^2\theta$ in place of $\sin^2\theta + \cos^2\theta = 1$. ■

EXAMPLE 4 Calculate $\sin\left[2\cos^{-1}\left(\frac{2}{3}\right)\right]$.

SOLUTION Recall the double-angle identity $\sin 2\theta = 2\sin\theta\cos\theta$. Thus,

$$\sin\left[2\cos^{-1}\left(\frac{2}{3}\right)\right] = 2\sin\left[\cos^{-1}\left(\frac{2}{3}\right)\right]\cos\left[\cos^{-1}\left(\frac{2}{3}\right)\right]$$

$$= 2\cdot\sqrt{1 - \left(\frac{2}{3}\right)^2}\cdot\frac{2}{3} = \frac{4\sqrt{5}}{9}$$ ■

Derivatives of Trigonometric Functions We learned in Section 2.4 the derivative formulas for the six trigonometric functions. They should be memorized.

$D_x \sin x = \cos x$	$D_x \cos x = -\sin x$
$D_x \tan x = \sec^2 x$	$D_x \cot x = -\csc^2 x$
$D_x \sec x = \sec x \tan x$	$D_x \csc x = -\csc x \cot x$

We can combine the rules above with the Chain Rule. For example, if $u = f(x)$ is differentiable, then

$$D_x \sin u = \cos u \cdot D_x u$$

Inverse Trigonometric Functions From the Inverse Function Theorem (Theorem 6.2B), we conclude that $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, and $\sec^{-1}$ are differentiable. Our aim is to find formulas for their derivatives. We state the results and then show how they can be derived.

Theorem B **Derivatives of Four Inverse Trigonometric Functions**

(i) $D_x \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1$

(ii) $D_x \cos^{-1} x = -\dfrac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1$

(iii) $D_x \tan^{-1} x = \dfrac{1}{1 + x^2}$

(iv) $D_x \sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2 - 1}}, \qquad |x| > 1$

Proof Our proofs follow the same pattern in each case. To prove (i), let $y = \sin^{-1} x$, so that

$$x = \sin y$$

Now differentiate both sides with respect to x, using the Chain Rule on the right-hand side. Then

$$1 = \cos y \, D_x y = \cos(\sin^{-1} x) \, D_x(\sin^{-1} x)$$
$$= \sqrt{1 - x^2} \, D_x(\sin^{-1} x)$$

At the last step, we used Theorem A(ii). We conclude that $D_x(\sin^{-1} x) = 1/\sqrt{1 - x^2}$.

Results (ii), (iii), and (iv) are proved similarly, but (iv) has a little twist. Let $y = \sec^{-1} x$, so

$$x = \sec y$$

Differentiating both sides with respect to x and using Theorem A(iv), we obtain

$$1 = \sec y \tan y \, D_x y$$
$$= \sec(\sec^{-1} x) \tan(\sec^{-1} x) \, D_x(\sec^{-1} x)$$
$$= \begin{cases} x\sqrt{x^2 - 1} \, D_x(\sec^{-1} x), & \text{if } x \geq 1 \\ x\left(-\sqrt{x^2 - 1}\right) D_x(\sec^{-1} x), & \text{if } x \leq -1 \end{cases}$$
$$= |x|\sqrt{x^2 - 1} \, D_x(\sec^{-1} x)$$

The desired result follows immediately. ■

$D_x \sec^{-1} x$

Here is another way to derive the formula for the derivative of $\sec^{-1} x$.

$$D_x \sec^{-1} x = D_x \cos^{-1}\left(\frac{1}{x}\right)$$
$$= \frac{-1}{\sqrt{1 - 1/x^2}} \cdot \frac{-1}{x^2}$$
$$= \frac{1}{\sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2}}{x^2}$$
$$= \frac{1}{\sqrt{x^2 - 1}} \cdot \frac{|x|}{x^2}$$
$$= \frac{1}{|x|\sqrt{x^2 - 1}}$$

EXAMPLE 5 Find $D_x \sin^{-1}(3x - 1)$.

SOLUTION We use Theorem B(i) and the Chain Rule.

$$D_x \sin^{-1}(3x - 1) = \frac{1}{\sqrt{1 - (3x - 1)^2}} D_x(3x - 1)$$
$$= \frac{3}{\sqrt{-9x^2 + 6x}}$$ ■

Of course, every differentiation formula leads to an integration formula, a matter we will say much more about in the next chapter. In particular,

1. $\displaystyle\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C$

2. $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C$

These integration formulas can be generalized slightly (see Problems 81–84) to the following:

1′. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

2′. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

3′. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

EXAMPLE 6 Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$.

SOLUTION

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$ ■

EXAMPLE 7 Evaluate $\displaystyle\int \frac{3}{\sqrt{5-9x^2}}\,dx$.

SOLUTION Think of $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}}$. Let $u = 3x$, so $du = 3\,dx$. Then

$$\int \frac{3}{\sqrt{5-9x^2}}\,dx = \int \frac{1}{\sqrt{5-u^2}}\,du = \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$
$$= \sin^{-1}\left(\frac{3x}{\sqrt{5}}\right) + C$$ ■

EXAMPLE 8 Evaluate $\displaystyle\int \frac{e^x}{4+9e^{2x}}\,dx$.

SOLUTION Think of $\displaystyle\int \frac{1}{a^2+u^2}\,du$. Let $u = 3e^x$, so $du = 3e^x\,dx$. Then

$$\int \frac{e^x}{4+9e^{2x}}\,dx = \frac{1}{3}\int \frac{1}{4+9e^{2x}}(3e^x\,dx) = \frac{1}{3}\int \frac{1}{4+u^2}\,du$$
$$= \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{3e^x}{2}\right) + C$$ ■

EXAMPLE 9 Evaluate $\displaystyle\int_6^{18} \frac{1}{x\sqrt{x^2-9}}\,dx$

SOLUTION

$$\int_6^{18} \frac{1}{x\sqrt{x^2-9}}\,dx = \frac{1}{3}\left[\sec^{-1}\frac{|x|}{3}\right]_6^{18}$$
$$= \frac{1}{3}\left(\sec^{-1}\frac{|18|}{3} - \sec^{-1}\frac{|6|}{3}\right)$$
$$= \frac{1}{3}\left(\sec^{-1} 6 - \frac{\pi}{3}\right) \approx 0.1187$$ ■

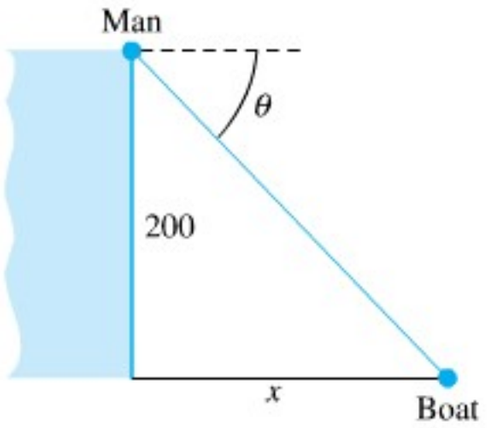


Figure 8

EXAMPLE 10 A man standing on top of a vertical cliff is 200 feet above a lake. As he watches, a motorboat moves directly away from the foot of the cliff at a rate of 25 feet per second. How fast is the angle of depression of his line of sight changing when the boat is 150 feet from the foot of the cliff?

SOLUTION The essential details are shown in Figure 8. Note that θ, the angle of depression, is

$$\theta = \tan^{-1}\left(\frac{200}{x}\right)$$

Thus,

$$\frac{d\theta}{dt} = \frac{1}{1 + (200/x)^2} \cdot \frac{-200}{x^2} \cdot \frac{dx}{dt} = \frac{-200}{x^2 + 40{,}000} \cdot \frac{dx}{dt}$$

When we substitute $x = 150$ and $dx/dt = 25$, we obtain $d\theta/dt = -0.08$ radian per second. ■

Manipulating the Integrand Before you make a substitution, you may find it helpful to rewrite the integrand in a more convenient form. Integrals with quadratic expressions in the denominator can often be reduced to standard forms by *completing the square*. Recall that $x^2 + bx$ becomes a perfect square by the addition of $(b/2)^2$.

EXAMPLE 11 Evaluate $\displaystyle\int \frac{7}{x^2 - 6x + 25}\,dx$.

SOLUTION

$$\begin{aligned}\int \frac{7}{x^2 - 6x + 25}\,dx &= \int \frac{7}{x^2 - 6x + 9 + 16}\,dx \\ &= 7\int \frac{1}{(x-3)^2 + 4^2}\,dx \\ &= \frac{7}{4}\tan^{-1}\left(\frac{x-3}{4}\right) + C\end{aligned}$$

We made the mental substitution $u = x - 3$ at the final stage. ■

Concepts Review

1. To obtain an inverse for the sine function, we restrict its domain to _____. The resulting inverse function is denoted by $\sin^{-1}$ or by _____.

2. To obtain an inverse for the tangent function, we restrict the domain to _____. The resulting inverse function is denoted by $\tan^{-1}$ or by _____.

3. $D_x \sin(\arcsin x) =$ _____.

4. Since $D_x \arctan x = 1/(1 + x^2)$, it follows that $4\displaystyle\int_0^1 1/(1 + x^2)\,dx =$ _____.

Problem Set 6.8

In Problems 1–10, find the exact value without using a calculator.

1. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

3. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

4. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

5. $\arctan\left(\sqrt{3}\right)$

6. $\operatorname{arcsec}(2)$

7. $\arcsin\left(-\frac{1}{2}\right)$

8. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

9. $\sin(\sin^{-1} 0.4567)$

10. $\cos(\sin^{-1} 0.56)$

In Problems 11–18, approximate each value.

11. $\sin^{-1}(0.1113)$

12. $\arccos(0.6341)$

13. $\cos(\text{arccot } 3.212)$

14. $\sec(\arccos 0.5111)$

15. $\sec^{-1}(-2.222)$

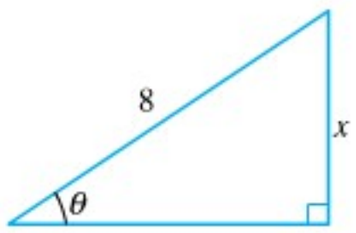
16. $\tan^{-1}(-60.11)$

17. $\cos(\sin(\tan^{-1} 2.001))$

18. $\sin^2(\ln(\cos 0.5555))$

In Problems 19–24, express θ in terms of x using the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, *and* $\sec^{-1}$.

19.



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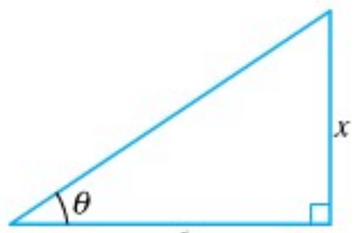

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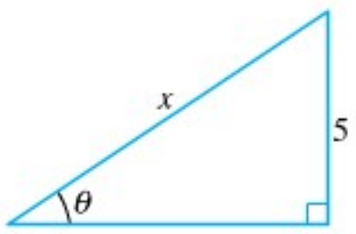

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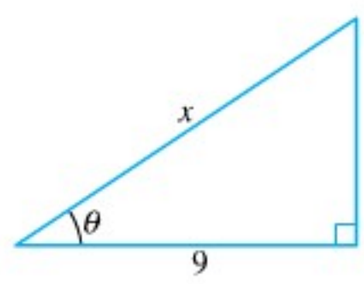

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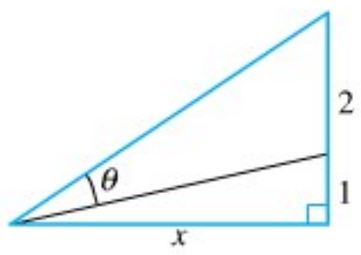

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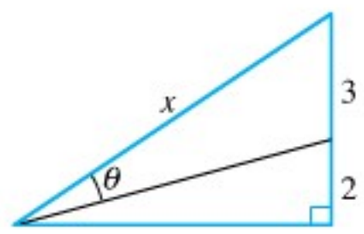


In Problems 25–28, find each value without using a calculator (see Example 4).

25. $\cos\left[2\sin^{-1}\left(-\frac{2}{3}\right)\right]$

26. $\tan\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]$

27. $\sin\left[\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right]$

28. $\cos\left[\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right]$

In Problems 29–32, show that each equation is an identity.

29. $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1 - x^2}}$

30. $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1 + x^2}}$

31. $\cos(2\sin^{-1} x) = 1 - 2x^2$

32. $\tan(2\tan^{-1} x) = \dfrac{2x}{1 - x^2}$

33. Find each limit.

(a) $\lim\limits_{x \to \infty} \tan^{-1} x$ (b) $\lim\limits_{x \to -\infty} \tan^{-1} x$

34. Find each limit.

(a) $\lim\limits_{x \to \infty} \sec^{-1} x$ (b) $\lim\limits_{x \to -\infty} \sec^{-1} x$

35. Find each limit.

(a) $\lim\limits_{x \to 1^-} \sin^{-1} x$ (b) $\lim\limits_{x \to -1^+} \sin^{-1} x$

36. Does $\lim\limits_{x \to 1} \sin^{-1} x$ exist? Explain.

37. Describe what happens to the slope of the tangent line to the graph of $y = \sin^{-1} x$ at the point c if c approaches 1 from the left.

38. Sketch the graph of $y = \cot^{-1} x$, assuming that it has been obtained by restricting the domain of the cotangent to $(0, \pi)$.

In Problems 39–54, find dy/dx.

39. $y = \ln(2 + \sin x)$

40. $y = e^{\tan x}$

41. $y = \ln(\sec x + \tan x)$

42. $y = -\ln(\csc x + \cot x)$

43. $y = \sin^{-1}(2x^2)$

44. $y = \arccos(e^x)$

45. $y = x^3 \tan^{-1}(e^x)$

46. $y = e^x \arcsin x^2$

47. $y = (\tan^{-1} x)^3$

48. $y = \tan(\cos^{-1} x)$

49. $y = \sec^{-1}(x^3)$

50. $y = (\sec^{-1} x)^3$

51. $y = (1 + \sin^{-1} x)^3$

52. $y = \sin^{-1}\left(\dfrac{1}{x^2 + 4}\right)$

53. $y = \tan^{-1}(\ln x^2)$

54. $y = x \operatorname{arcsec}(x^2 + 1)$

In Problems 55–72, evaluate each integral.

55. $\displaystyle\int \cos 3x \, dx$

56. $\displaystyle\int x \sin(x^2) \, dx$

57. $\displaystyle\int \sin 2x \cos 2x \, dx$

58. $\displaystyle\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

59. $\displaystyle\int_0^1 e^{2x} \cos(e^{2x}) \, dx$

60. $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x \, dx$

61. $\displaystyle\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1 - x^2}} \, dx$

62. $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2 - 1}}$

63. $\displaystyle\int_{-1}^1 \frac{1}{1 + x^2} \, dx$

64. $\displaystyle\int_0^{\pi/2} \frac{\sin\theta}{1 + \cos^2\theta} \, d\theta$

65. $\displaystyle\int \frac{1}{1 + 4x^2} \, dx$

66. $\displaystyle\int \frac{e^x}{1 + e^{2x}} \, dx$

67. $\displaystyle\int \frac{1}{\sqrt{12 - 9x^2}} \, dx$

68. $\displaystyle\int \frac{x}{\sqrt{12 - 9x^2}} \, dx$

69. $\displaystyle\int \frac{1}{x^2 - 6x + 13} \, dx$

70. $\displaystyle\int \frac{1}{2x^2 + 8x + 25} \, dx$

71. $\displaystyle\int \frac{1}{x\sqrt{4x^2 - 9}} \, dx$

72. $\displaystyle\int \frac{x + 1}{\sqrt{4 - 9x^2}} \, dx$

C **73.** A picture 5 feet in height is hung on a wall so that its bottom is 8 feet from the floor, as shown in Figure 9. A viewer with eye level at 5.4 feet stands b feet from the wall. Express θ, the vertical angle subtended by the picture at her eye, in terms of b, and then find θ if $b = 12.9$ feet.

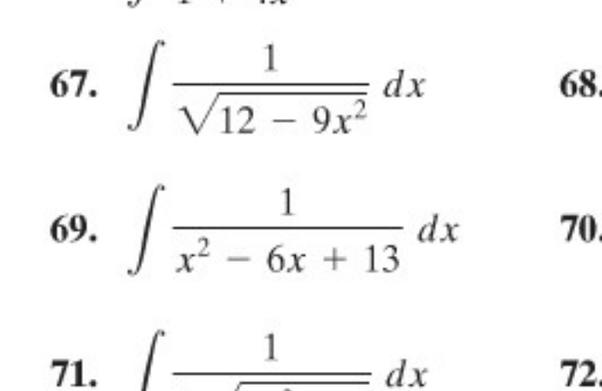


Figure 9

74. Find formulas for $f^{-1}(x)$ for each of the following functions f, first indicating how you would restrict the domain so that f has an inverse. For example, if $f(x) = 3 \sin 2x$ and we restrict the domain to $-\pi/4 \le x \le \pi/4$, then $f^{-1}(x) = \frac{1}{2}\sin^{-1}(x/3)$.

(a) $f(x) = 3 \cos 2x$ (b) $f(x) = 2 \sin 3x$

(c) $f(x) = \frac{1}{2} \tan x$ (d) $f(x) = \sin \dfrac{1}{x}$

75. By repeated use of the addition formula

$$\tan(x + y) = (\tan x + \tan y)/(1 - \tan x \tan y)$$

show that

$$\frac{\pi}{4} = 3 \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{5}{99}\right)$$

76. Verify that

$$\frac{\pi}{4} = 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

a result discovered by John Machin in 1706 and used by him to calculate the first 100 decimal places of π.

77. Without using calculus, find a formula for the area of the shaded region in Figure 10 in terms of a and b. Note that the center of the larger circle is on the rim of the smaller.

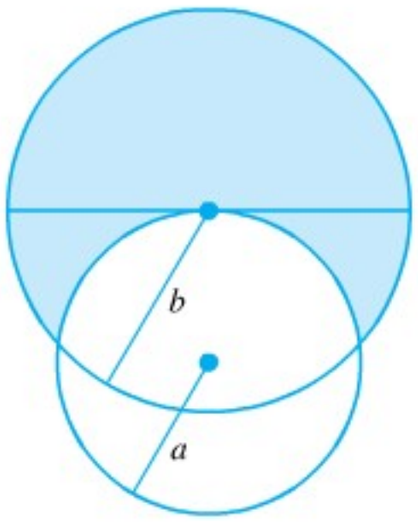


Figure 10

GC **78.** Draw the graphs of

$$y = \arcsin x \quad \text{and} \quad y = \arctan\left(x/\sqrt{1 - x^2}\right)$$

using the same axes. Make a conjecture. Prove it.

GC **79.** Draw the graph of $y = \pi/2 - \arcsin x$. Make a conjecture. Prove it.

GC **80.** Draw the graph of $y = \sin(\arcsin x)$ on $[-1, 1]$. Then draw the graph of $y = \arcsin(\sin x)$ on $[-2\pi, 2\pi]$. Explain the differences that you observe.

81. Show that

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C, \qquad a > 0$$

by writing $a^2 - x^2 = a^2[1 - (x/a)^2]$ and making the substitution $u = x/a$.

82. Show the result in Problem 81 by differentiating the right side to get the integrand.

83. Show that

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \neq 0$$

84. Show that

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|x|}{a} + C, \qquad a > 0$$

85. Show, by differentiating the right side, that

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C, \qquad a > 0$$

86. Use the result of Problem 85 to show that

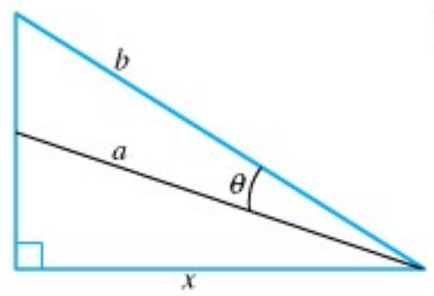
$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{2}$$

Why is this result expected?

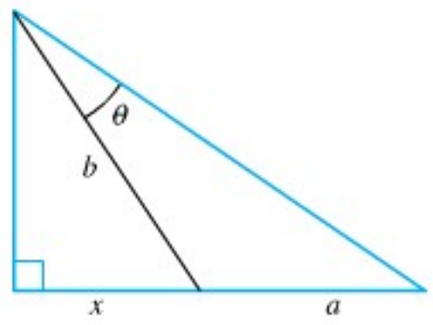
87. The lower edge of a wall hanging, 10 feet in height, is 2 feet above the observer's eye level. Find the ideal distance b to stand from the wall for viewing the hanging; that is, find b that maximizes the angle subtended at the viewer's eye. (See Problem 73.)

88. Express $d\theta/dt$ in terms of x, dx/dt, and the constants a and b.

(a)



(b)



89. The structural steel work of a new office building is finished. Across the street, 60 feet from the ground floor of the freight elevator shaft in the building, a spectator is standing and watching the freight elevator ascend at a constant rate of 15 feet per second. How fast is the angle of elevation of the spectator's line of sight to the elevator increasing 6 seconds after his line of sight passes the horizontal?

90. An airplane is flying at a constant altitude of 2 miles and a constant speed of 600 miles per hour on a straight course that will take it directly over an observer on the ground. How fast is the angle of elevation of the observer's line of sight increasing when the distance from her to the plane is 3 miles? Give your result in radians per minute.

91. A revolving beacon light is located on an island and is 2 miles away from the nearest point P of the straight shoreline of the mainland. The beacon throws a spot of light that moves along the shoreline as the beacon revolves. If the speed of the spot of light on the shoreline is 5π miles per minute when the spot is 1 mile from P, how fast is the beacon revolving?

92. A man on a dock is pulling in a rope attached to a rowboat at a rate of 5 feet per second. If the man's hands are 8 feet higher than the point where the rope is attached to the boat, how fast is the angle of depression of the rope changing when there are still 17 feet of rope out?

C **93.** A visitor from outer space is approaching the earth (radius = 6376 kilometers) at 2 kilometers per second. How fast is the angle θ subtended by the earth at her eye increasing when she is 3000 kilometers from the surface?

Answers to Concepts Review: **1.** $[-\pi/2, \pi/2]$; arcsin **2.** $(-\pi/2, \pi/2)$; arctan **3.** 1 **4.** π

6.9 The Hyperbolic Functions and Their Inverses

In both mathematics and science, certain combinations of e^x and e^{-x} occur so often that they are given special names.

Definition Hyperbolic Functions

The hyperbolic sine, hyperbolic cosine, and four related functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

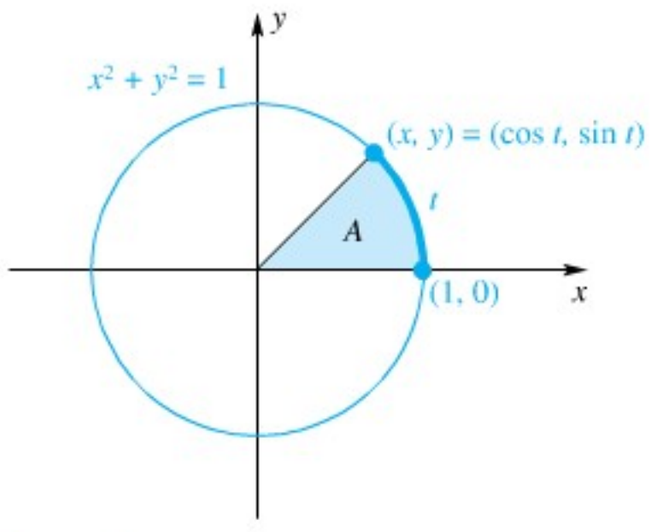


Figure 1

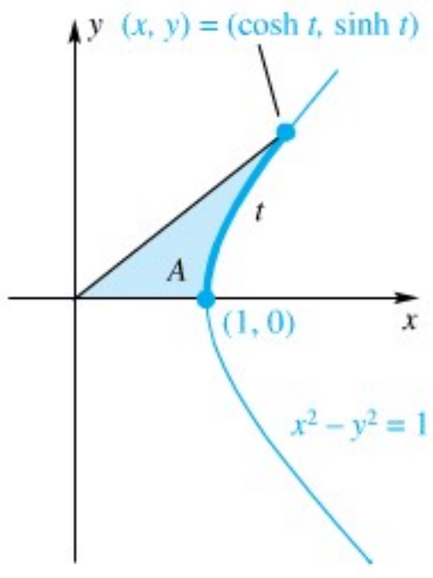


Figure 2

The terminology suggests that there must be some connection with the trigonometric functions; there is. First, the fundamental identity for the hyperbolic functions (reminiscent of $\cos^2 x + \sin^2 x = 1$ in trigonometry) is

$$\cosh^2 x - \sinh^2 x = 1$$

To verify it, we write

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1$$

Second, recall that the trigonometric functions are intimately related to the unit circle (Figure 1), so much so that they are sometimes called the *circular functions*. In fact, the parametric equations $x = \cos t$, $y = \sin t$ describe the unit circle. In parallel fashion, the parametric equations $x = \cosh t$, $y = \sinh t$ describe the right branch of the unit hyperbola $x^2 - y^2 = 1$ (Figure 2). Moreover, in both cases the parameter t is related to the shaded area A by $t = 2A$, though this is not obvious in the second case (Problem 56).

Since $\sinh(-x) = -\sinh x$, sinh is an odd function; $\cosh(-x) = \cosh x$, so cosh is an even function. Correspondingly, the graph of $y = \sinh x$ is symmetric with respect to the origin and the graph of $y = \cosh x$ is symmetric with respect to the y-axis. Similarly, tanh is an odd function and sech is an even function. The graphs are shown in Figure 3.

Derivatives of Hyperbolic Functions We can find $D_x \sinh x$ and $D_x \cosh x$ directly from the definitions.

$$D_x \sinh x = D_x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$D_x \cosh x = D_x\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

Note that these facts confirm the character of the graphs in Figure 3. For example, since $D_x(\sinh x) = \cosh x > 0$, the graph of hyperbolic sine is always increasing. Similarly, $D_x^2(\cosh x) = \cosh x > 0$, which means that the graph of hyperbolic cosine is concave upward.

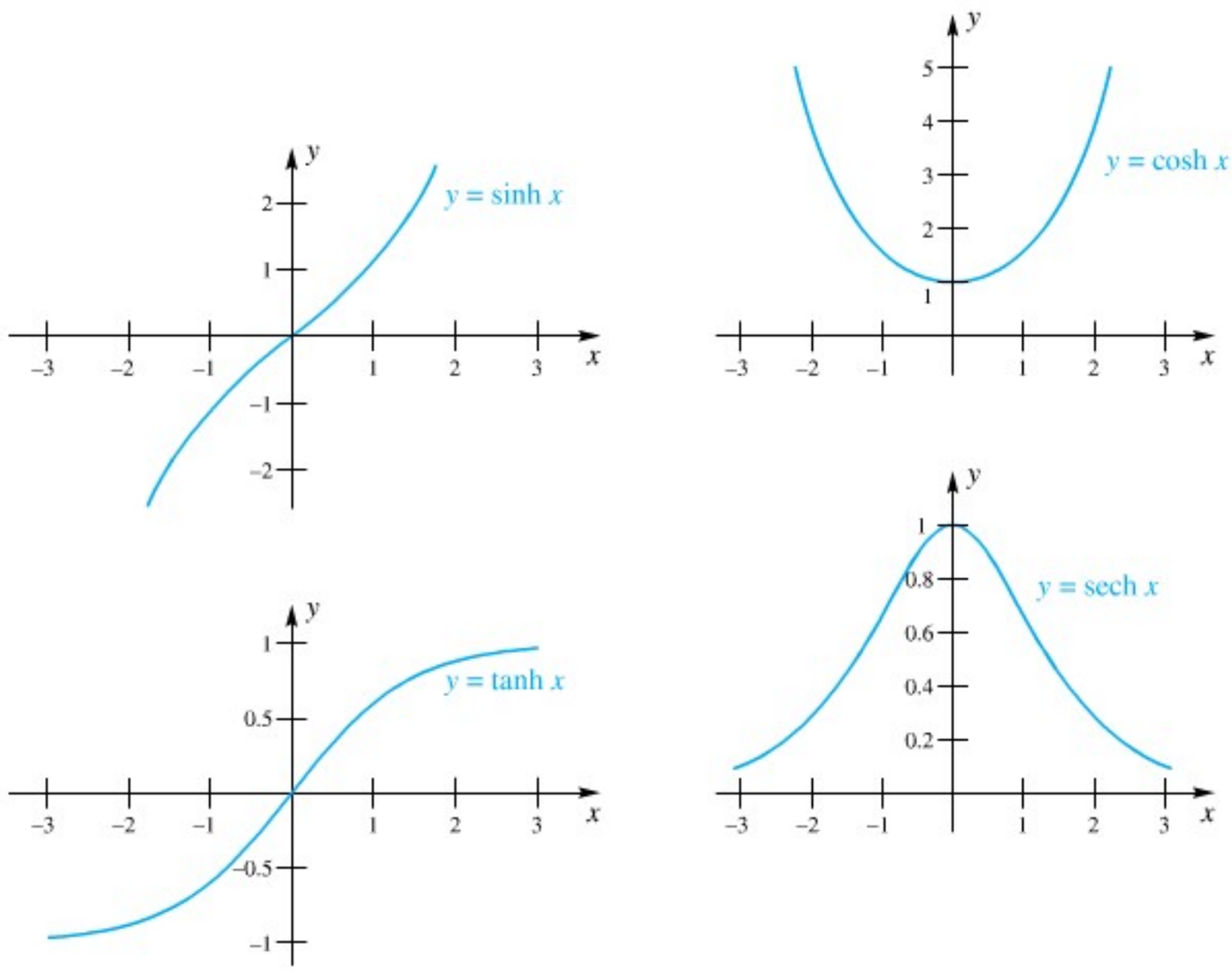


Figure 3

The derivatives of the other four hyperbolic functions follow from those for the first two, combined with the Quotient Rule. The results are summarized in Theorem A.

Theorem A **Derivatives of Hyperbolic Functions**

$$D_x \sinh x = \cosh x \qquad D_x \cosh x = \sinh x$$

$$D_x \tanh x = \operatorname{sech}^2 x \qquad D_x \coth x = -\operatorname{csch}^2 x$$

$$D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x \qquad D_x \operatorname{csch} x = -\operatorname{csch} x \coth x$$

Another way that the trigonometric and hyperbolic functions are connected concerns differential equations. The functions $\sin x$ and $\cos x$ are solutions to the second-order differential equation $y'' = -y$, and $\sinh x$ and $\cosh x$ are solutions to the differential equation $y'' = y$.

EXAMPLE 1 Find $D_x \tanh(\sin x)$.

SOLUTION

$$\begin{aligned} D_x \tanh(\sin x) &= \operatorname{sech}^2(\sin x)\, D_x(\sin x) \\ &= \cos x \cdot \operatorname{sech}^2(\sin x) \end{aligned}$$

■

EXAMPLE 2 Find $D_x \cosh^2(3x - 1)$.

SOLUTION We apply the Chain Rule twice.

$$\begin{aligned} D_x \cosh^2(3x - 1) &= 2\cosh(3x - 1)\, D_x \cosh(3x - 1) \\ &= 2\cosh(3x - 1)\sinh(3x - 1)\, D_x(3x - 1) \\ &= 6\cosh(3x - 1)\sinh(3x - 1) \end{aligned}$$

■

EXAMPLE 3 Find $\int \tanh x\, dx$.

SOLUTION Let $u = \cosh x$, so $du = \sinh x\, dx$.

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx = \int \frac{1}{u}\, du$$
$$= \ln|u| + C = \ln|\cosh x| + C = \ln(\cosh x) + C$$

We could drop the absolute value signs because $\cosh x > 0$. ■

Inverse Hyperbolic Functions Since hyperbolic sine and hyperbolic tangent have positive derivatives, they are increasing functions and automatically have inverses. To obtain inverses for hyperbolic cosine and hyperbolic secant, we restrict their domains to $x \geq 0$. Thus,

$$x = \sinh^{-1} y \quad \Leftrightarrow \quad y = \sinh x$$
$$x = \cosh^{-1} y \quad \Leftrightarrow \quad y = \cosh x \quad \text{and} \quad x \geq 0$$
$$x = \tanh^{-1} y \quad \Leftrightarrow \quad y = \tanh x$$
$$x = \operatorname{sech}^{-1} y \quad \Leftrightarrow \quad y = \operatorname{sech} x \quad \text{and} \quad x \geq 0$$

Since the hyperbolic functions are defined in terms of e^x and e^{-x}, it is not too surprising that the inverse hyperbolic functions can be expressed in terms of the natural logarithm. For example, consider $y = \cosh x$ for $x \geq 0$; that is, consider

$$y = \frac{e^x + e^{-x}}{2}, \qquad x \geq 0$$

Our goal is to solve this equation for x, which will give $\cosh^{-1} y$. Multiplying both sides by $2e^x$, we get $2ye^x = e^{2x} + 1$, or

$$(e^x)^2 - 2ye^x + 1 = 0, \qquad x \geq 0$$

If we solve this quadratic equation in e^x, we obtain

$$e^x = \frac{2y + \sqrt{(2y)^2 - 4}}{2} = y + \sqrt{y^2 - 1}$$

The Quadratic Formula gives two solutions, the one given above and $\left(2y - \sqrt{(2y)^2 - 4}\right)/2$. This latter solution is extraneous because it is less than 1, whereas e^x is greater than 1 for all $x > 0$. Thus, $x = \ln\left(y + \sqrt{y^2 - 1}\right)$, so

$$x = \cosh^{-1} y = \ln\left(y + \sqrt{y^2 - 1}\right)$$

Similar arguments apply to each of the inverse hyperbolic functions. We obtain the following results (note that the roles of x and y have been interchanged). Figure 3 suggests the necessary domain restrictions. Graphs of the inverse hyperbolic functions are shown in Figure 4.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$
$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \qquad x \geq 1$$
$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1 + x}{1 - x}, \qquad -1 < x < 1$$
$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), \qquad 0 < x \leq 1$$

Each of these functions is differentiable. In fact,

$$D_x \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$$

$$D_x \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}, \qquad x > 1$$

$$D_x \tanh^{-1} x = \frac{1}{1 - x^2}, \qquad -1 < x < 1$$

$$D_x \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1 - x^2}}, \qquad 0 < x < 1$$

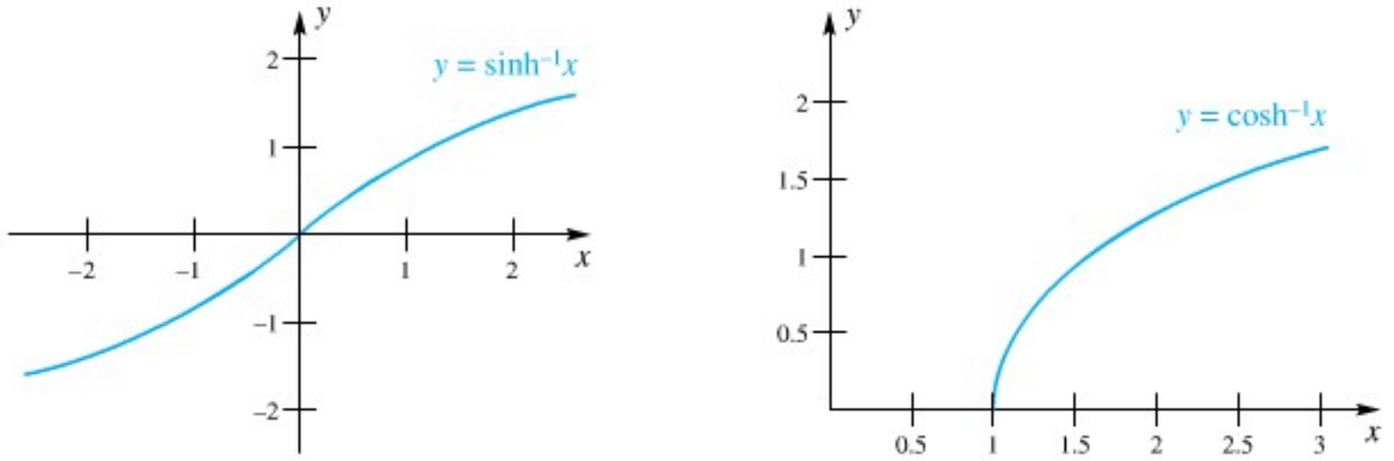


Figure 4

EXAMPLE 4 Show that $D_x \sinh^{-1} x = 1/\sqrt{x^2 + 1}$ by two different methods.

SOLUTION

Method 1 Let $y = \sinh^{-1} x$, so

$$x = \sinh y$$

Now differentiate both sides with respect to x.

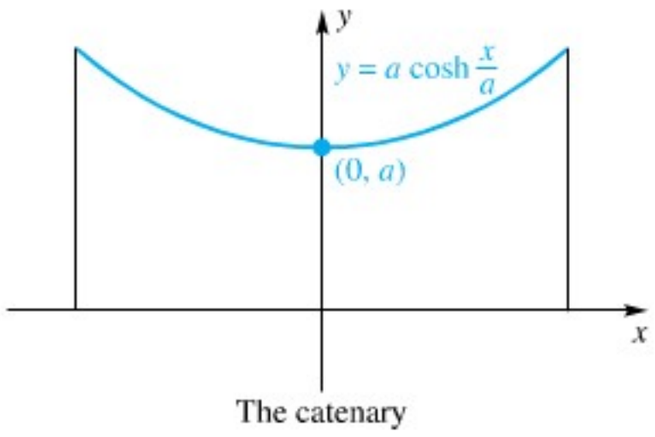
$$1 = (\cosh y)\, D_x y$$

Thus,

$$D_x y = D_x(\sinh^{-1} x) = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

Method 2 Use the logarithmic expression for $\sinh^{-1} x$.

$$\begin{aligned} D_x(\sinh^{-1} x) &= D_x \ln\left(x + \sqrt{x^2 + 1}\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} D_x\left(x + \sqrt{x^2 + 1}\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$



The catenary

Figure 5

An Inverted Catenary

Applications: The Catenary If a homogeneous flexible cable or chain is suspended between two fixed points at the same height, it forms a curve called a **catenary** (Figure 5). Furthermore (see Problem 53), a catenary can be placed in a coordinate system so that its equation takes the form

$$y = a \cosh \frac{x}{a}$$

EXAMPLE 5 Find the length of the catenary $y = a \cosh(x/a)$ between $x = -a$ and $x = a$.

SOLUTION The desired length (see Section 5.4) is given by

$$\begin{aligned}
\int_{-a}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_{-a}^{a} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\, dx \\
&= \int_{-a}^{a} \sqrt{\cosh^2\left(\frac{x}{a}\right)}\, dx \\
&= 2\int_{0}^{a} \cosh\left(\frac{x}{a}\right) dx \\
&= 2a\int_{0}^{a} \cosh\left(\frac{x}{a}\right)\left(\frac{1}{a}\, dx\right) \\
&= \left[2a \sinh \frac{x}{a}\right]_0^a \\
&= 2a \sinh 1 \approx 2.35a
\end{aligned}$$

■

Concepts Review

1. sinh and cosh are defined by $\sinh x =$ _____ and $\cosh x =$ _____.

2. In *hyperbolic* trigonometry, the identity corresponding to $\sin^2 x + \cos^2 x = 1$ is _____.

3. Because of the identity in Question 2, the graph of the parametric equations $x = \cosh t$, $y = \sinh t$ is _____.

4. The graph of $y = a \cosh(x/a)$ is a curve called a _____; this curve is important as a model for _____.

Problem Set 6.9

In Problems 1–12, verify that the given equations are identities.

1. $e^x = \cosh x + \sinh x$

2. $e^{2x} = \cosh 2x + \sinh 2x$

3. $e^{-x} = \cosh x - \sinh x$

4. $e^{-2x} = \cosh 2x - \sinh 2x$

5. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

6. $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

7. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

8. $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

9. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

10. $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

11. $\sinh 2x = 2 \sinh x \cosh x$

12. $\cosh 2x = \cosh^2 x + \sinh^2 x$

In Problems 13–36, find $D_x y$.

13. $y = \sinh^2 x$

14. $y = \cosh^2 x$

15. $y = 5 \sinh^2 x$

16. $y = \cosh^3 x$

17. $y = \cosh(3x + 1)$

18. $y = \sinh(x^2 + x)$

19. $y = \ln(\sinh x)$

20. $y = \ln(\coth x)$

21. $y = x^2 \cosh x$

22. $y = x^{-2} \sinh x$

23. $y = \cosh 3x \sinh x$

24. $y = \sinh x \cosh 4x$

25. $y = \tanh x \sinh 2x$

26. $y = \coth 4x \sinh x$

27. $y = \sinh^{-1}(x^2)$

28. $y = \cosh^{-1}(x^3)$

29. $y = \tanh^{-1}(2x - 3)$

30. $y = \coth^{-1}(x^5)$

31. $y = x \cosh^{-1}(3x)$

32. $y = x^2 \sinh^{-1}(x^5)$

33. $y = \ln(\cosh^{-1} x)$

34. $y = \cosh^{-1}(\cos x)$

35. $y = \tanh(\cot x)$ **36.** $y = \coth^{-1}(\tanh x)$

37. Find the area of the region bounded by $y = \cosh 2x$, $y = 0$, $x = 0$, and $x = \ln 3$.

In Problems 38–45, evaluate each integral.

38. $\int \sinh(3x + 2)\,dx$ **39.** $\int x \cosh(\pi x^2 + 5)\,dx$

40. $\int \frac{\cosh \sqrt{z}}{\sqrt{z}}\,dz$ **41.** $\int \frac{\sinh(2z^{1/4})}{\sqrt[4]{z^3}}\,dz$

42. $\int e^x \sinh e^x\,dx$ **43.** $\int \cos x \sinh(\sin x)\,dx$

44. $\int \tanh x \ln(\cosh x)\,dx$

45. $\int x \coth x^2 \ln(\sinh x^2)\,dx$

46. Find the area of the region bounded by $y = \cosh 2x$, $y = 0$, $x = -\ln 5$, and $x = \ln 5$.

47. Find the area of the region bounded by $y = \sinh x$, $y = 0$, and $x = \ln 2$.

48. Find the area of the region bounded by $y = \tanh x$, $y = 0$, $x = -8$, and $x = 8$.

49. The region bounded by $y = \cosh x$, $y = 0$, $x = 0$, and $x = 1$ is revolved about the x-axis. Find the volume of the resulting solid. *Hint:* $\cosh^2 x = (1 + \cosh 2x)/2$.

50. The region bounded by $y = \sinh x$, $y = 0$, $x = 0$, and $x = \ln 10$ is revolved about the x-axis. Find the volume of the resulting solid.

51. The curve $y = \cosh x, 0 \le x \le 1$, is revolved about the x-axis. Find the area of the resulting surface.

52. The curve $y = \sinh x, 0 \le x \le 1$, is revolved about the x-axis. Find the area of the resulting surface.

53. To derive the equation of a hanging cable (catenary), we consider the section AP from the lowest point A to a general point $P(x, y)$ (see Figure 6) and imagine the rest of the cable to have been removed.

The forces acting on the cable are

1. H = horizontal tension pulling at A;
2. T = tangential tension pulling at P;
3. $W = \delta s$ = weight of s feet of cable of density δ pounds per foot.

To be in equilibrium, the horizontal and vertical components of T must just balance H and W, respectively. Thus, $T \cos\phi = H$ and $T \sin\phi = W = \delta s$, and so

$$\frac{T \sin\phi}{T \cos\phi} = \tan\phi = \frac{\delta s}{H}$$

But since $\tan\phi = dy/dx$, we get

$$\frac{dy}{dx} = \frac{\delta s}{H}$$

and therefore

$$\frac{d^2y}{dx^2} = \frac{\delta}{H}\frac{ds}{dx} = \frac{\delta}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Now show that $y = a\cosh(x/a) + C$ satisfies this differential equation with $a = H/\delta$.

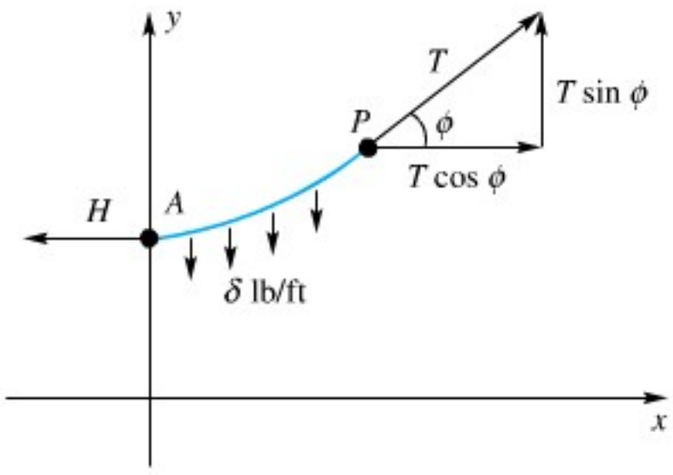


Figure 6

C **54.** Call the graph of $y = b - a\cosh(x/a)$ an inverted catenary and imagine it to be an arch sitting on the x-axis. Show that if the width of this arch along the x-axis is $2a$ then each of the following is true.

(a) $b = a\cosh 1 \approx 1.54308a$.

(b) The height of the arch is approximately $0.54308a$.

(c) The height of an arch of width 48 is approximately 13.

C **55.** A farmer built a large hayshed of length 100 feet and width 48 feet. A cross section has the shape of an inverted catenary (see Problem 54) with equation $y = 37 - 24\cosh(x/24)$.

(a) Draw a picture of this shed.

(b) Find the volume of the shed.

(c) Find the surface area of the roof of the shed.

56. Show that $A = t/2$, where A denotes the area in Figure 2 of this section. *Hint:* At some point you will need to use Formula 44 from the back of the book.

57. Demonstrate that for every real number r:

(a) $(\sinh x + \cosh x)^r = \sinh rx + \cosh rx$

(b) $(\cosh x - \sinh x)^r = \cosh rx - \sinh rx$

(c) $(\cos x + i \sin x)^r = \cos rx + i \sin rx$

(d) $(\cos x - i \sin x)^r = \cos rx - i \sin rx$

58. The **gudermannian** of t is defined by

$$\text{gd}(t) = \tan^{-1}(\sinh t)$$

Show that

(a) gd is odd and increasing with an inflection point at the origin;

(b) $\text{gd}(t) = \sin^{-1}(\tanh t) = \int_0^t \text{sech}\, u\,du$.

59. Show that the area under the curve $y = \cosh t$, $0 \le t \le x$, is numerically equal to its arc length.

60. Find the equation of the Gateway Arch in St. Louis, Missouri, given that it is an inverted catenary (see Problem 54). Assume that it stands on the x-axis, that it is symmetric with respect to the y-axis, and that it is 630 feet wide at the base and 630 feet high at the center.

GC **61.** Draw the graphs of $y = \sinh x$, $y = \ln\left(x + \sqrt{x^2 + 1}\right)$, and $y = x$ using the same axes and scaled so that $-3 \le x \le 3$ and $-3 \le y \le 3$. What does this demonstrate?

CHAPTER 7

Techniques of Integration

7.1 Basic Integration Rules
7.2 Integration by Parts
7.3 Some Trigonometric Integrals
7.4 Rationalizing Substitutions
7.5 Integration of Rational Functions Using Partial Fractions
7.6 Strategies for Integration

7.1 Basic Integration Rules

Our repertoire of functions now includes all the elementary functions. These are the constant functions, the power functions, the algebraic functions, the logarithmic and exponential functions, the trigonometric and inverse trigonometric functions, and all functions obtained from them by addition, subtraction, multiplication, division, and composition. Thus,

$$f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$g(x) = (1 + \cos^4 x)^{1/2}$$

$$h(x) = \frac{3^{x^2-2x}}{\ln(x^2 + 1)} - \sin[\cos(\cosh x)]$$

are elementary functions.

Differentiation of an elementary function is straightforward, requiring only a systematic use of the rules that we have learned. And the result is always an elementary function. Integration (antidifferentiation) is a far different matter. It involves a few techniques and a large bag of tricks; what is worse, it does not always yield an elementary function. For example, it is known that the antiderivatives of e^{-x^2} and $(\sin x)/x$ are not elementary functions.

The two principal techniques for integration are *substitution* and *integration by parts*. The method of substitution was introduced in Section 4.4; we have used it occasionally in the intervening chapters.

Standard Forms Effective use of the method of substitution and integration by parts depends on the ready availability of a list of known integrals. One such list (but too long to memorize) appears inside the back cover of this book. The short list shown below is so useful that we think that every calculus student should memorize it.

Standard Integral Forms

Constants, Powers

1. $\int k\,du = ku + C$

2. $\int u^r\,du = \begin{cases} \dfrac{u^{r+1}}{r+1} + C & r \neq -1 \\ \ln|u| + C & r = -1 \end{cases}$

Exponentials

3. $\int e^u\,du = e^u + C$

4. $\int a^u\,du = \dfrac{a^u}{\ln a} + C, a \neq 1, a > 0$

Trigonometric Functions

5. $\int \sin u\,du = -\cos u + C$

6. $\int \cos u\,du = \sin u + C$

7. $\int \sec^2 u\,du = \tan u + C$

8. $\int \csc^2 u\,du = -\cot u + C$

9. $\int \sec u \tan u\,du = \sec u + C$

10. $\int \csc u \cot u\,du = -\csc u + C$

11. $\int \tan u\,du = -\ln|\cos u| + C$

12. $\int \cot u\,du = \ln|\sin u| + C$

Algebraic Functions

13. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$

14. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$

15. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C = \frac{1}{a}\cos^{-1}\left(\frac{a}{|u|}\right) + C$

Hyperbolic Functions

16. $\displaystyle\int \sinh u\, du = \cosh u + C$

17. $\displaystyle\int \cosh u\, du = \sinh u + C$

Substitution in Indefinite Integrals Suppose that you face an indefinite integral. If it is a standard form, simply write the answer. If not, look for a substitution that will change it to a standard form. If the first substitution that you try does not work, try another. Skill at this, like most worthwhile activities, depends on practice.

The method of substitution was given in Theorem 4.4B and is restated here for easy reference.

Theorem A Substitution in Indefinite Integrals

Let g be a differentiable function and suppose that F is an antiderivative of f. Then, if $u = g(x)$,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

EXAMPLE 1 Find $\displaystyle\int \frac{x}{\cos^2(x^2)}dx$.

SOLUTION Look at this integral for a few moments. Since $1/\cos^2 x = \sec^2 x$, you may be reminded of the standard form $\int \sec^2 u\, du$. Let $u = x^2$, $du = 2x\,dx$. Then

$$\int \frac{x}{\cos^2(x^2)}\,dx = \frac{1}{2}\int \frac{1}{\cos^2(x^2)}\cdot 2x\,dx = \frac{1}{2}\int \sec^2 u\,du$$

$$= \frac{1}{2}\tan u + C = \frac{1}{2}\tan(x^2) + C$$

■

EXAMPLE 2 Find $\displaystyle\int \frac{3}{\sqrt{5 - 9x^2}}dx$.

SOLUTION Think of $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}}$. Let $u = 3x$, so $du = 3\,dx$. Then

$$\int \frac{3}{\sqrt{5 - 9x^2}}\,dx = \int \frac{1}{\sqrt{5 - u^2}}\,du = \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= \sin^{-1}\left(\frac{3x}{\sqrt{5}}\right) + C$$

■

EXAMPLE 3 Find $\displaystyle\int \frac{6e^{1/x}}{x^2}\,dx$.

SOLUTION Think of $\int e^u\,du$. Let $u = 1/x$, so $du = (-1/x^2)\,dx$. Then

$$\int \frac{6e^{1/x}}{x^2}\,dx = -6\int e^{1/x}\left(\frac{-1}{x^2}\,dx\right) = -6\int e^u\,du$$

$$= -6e^u + C = -6e^{1/x} + C$$

■

EXAMPLE 4 Find $\int \frac{e^x}{4 + 9e^{2x}}\,dx$.

SOLUTION Think of $\int \frac{1}{a^2 + u^2}\,du$. Let $u = 3e^x$, so $du = 3e^x\,dx$. Then

$$\int \frac{e^x}{4 + 9e^{2x}}\,dx = \frac{1}{3}\int \frac{1}{4 + 9e^{2x}}(3e^x\,dx) = \frac{1}{3}\int \frac{1}{4 + u^2}\,du$$

$$= \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{3e^x}{2}\right) + C$$ ■

No law says that you have to write out the u-substitution. If you can do the substitution mentally, that is fine. Here are two illustrations.

EXAMPLE 5 Find $\int x\cos x^2\,dx$

SOLUTION Mentally substitute $u = x^2$.

$$\int x\cos x^2\,dx = \frac{1}{2}\int (\cos x^2)(2x\,dx) = \frac{1}{2}\sin x^2 + C$$ ■

EXAMPLE 6 Find $\int \frac{a^{\tan t}}{\cos^2 t}\,dt$.

SOLUTION Mentally, substitute $u = \tan t$.

$$\int \frac{a^{\tan t}}{\cos^2 t}\,dt = \int a^{\tan t}(\sec^2 t\,dt) = \frac{a^{\tan t}}{\ln a} + C$$ ■

Substitution in Definite Integrals This topic was covered in Section 4.4. It is just like substitution in indefinite integrals, but we must remember to make the appropriate change in the limits of integration.

EXAMPLE 7 Evaluate $\int_2^5 t\sqrt{t^2 - 4}\,dt$.

SOLUTION Let $u = t^2 - 4$, so $du = 2t\,dt$; note that when $t = 2, u = 0$, and when $t = 5, u = 21$. Thus,

$$\int_2^5 t\sqrt{t^2 - 4}\,dt = \frac{1}{2}\int_2^5 (t^2 - 4)^{1/2}(2t\,dt)$$

$$= \frac{1}{2}\int_0^{21} u^{1/2}\,du$$

$$= \left[\frac{1}{3}u^{3/2}\right]_0^{21} = \frac{1}{3}(21)^{3/2} \approx 32.08$$ ■

EXAMPLE 8 Find $\int_1^3 x^3\sqrt{x^4 + 11}\,dx$.

SOLUTION Mentally substitute $u = x^4 + 11$.

$$\int_1^3 x^3\sqrt{x^4 + 11}\,dx = \frac{1}{4}\int_1^3 (x^4 + 11)^{1/2}(4x^3\,dx)$$

$$= \left[\frac{1}{6}(x^4 + 11)^{3/2}\right]_1^3$$

$$= \frac{1}{6}[92^{3/2} - 12^{3/2}] \approx 140.144$$ ■

Concepts Review

1. Differentiation of an elementary function is straightforward, but there are cases where the antiderivative of an elementary function cannot be expressed as a(an) _____.

2. The substitution $u = 1 + x^3$ transforms $\int 3x^2(1 + x^3)^5\,dx$ to _____.

3. The substitution $u =$ _____ transforms $\int e^x/(4 + e^{2x})\,dx$ to $\int 1/(4 + u^2)\,du$.

4. The substitution $u = 1 + \sin x$ transforms $\int_0^{\pi/2} (1 + \sin x)^3 \cos x\,dx$ to _____.

Problem Set 7.1

In Problems 1–54, perform the indicated integrations.

1. $\int (x - 2)^5\,dx$

2. $\int \sqrt{3x}\,dx$

3. $\int_0^2 x(x^2 + 1)^5\,dx$

4. $\int_0^1 x\sqrt{1 - x^2}\,dx$

5. $\int \dfrac{dx}{x^2 + 4}$

6. $\int \dfrac{e^x}{2 + e^x}\,dx$

7. $\int \dfrac{x}{x^2 + 4}\,dx$

8. $\int \dfrac{2t^2}{2t^2 + 1}\,dt$

9. $\int 6z\sqrt{4 + z^2}\,dz$

10. $\int \dfrac{5}{\sqrt{2t + 1}}\,dt$

11. $\int \dfrac{\tan z}{\cos^2 z}\,dz$

12. $\int e^{\cos z} \sin z\,dz$

13. $\int \dfrac{\sin\sqrt{t}}{\sqrt{t}}\,dt$

14. $\int \dfrac{2x\,dx}{\sqrt{1 - x^4}}$

15. $\int_0^{\pi/4} \dfrac{\cos x}{1 + \sin^2 x}\,dx$

16. $\int_0^{3/4} \dfrac{\sin\sqrt{1 - x}}{\sqrt{1 - x}}\,dx$

17. $\int \dfrac{3x^2 + 2x}{x + 1}\,dx$

18. $\int \dfrac{x^3 + 7x}{x - 1}\,dx$

19. $\int \dfrac{\sin(\ln 4x^2)}{x}\,dx$

20. $\int \dfrac{\sec^2(\ln x)}{2x}\,dx$

21. $\int \dfrac{6e^x}{\sqrt{1 - e^{2x}}}\,dx$

22. $\int \dfrac{x}{x^4 + 4}\,dx$

23. $\int \dfrac{3e^{2x}}{\sqrt{1 - e^{2x}}}\,dx$

24. $\int \dfrac{x^3}{x^4 + 4}\,dx$

25. $\int_0^1 t\,3^{t^2}\,dt$

26. $\int_0^{\pi/6} 2^{\cos x} \sin x\,dx$

27. $\int \dfrac{\sin x - \cos x}{\sin x}\,dx$

28. $\int \dfrac{\sin(4t - 1)}{1 - \sin^2(4t - 1)}\,dt$

29. $\int e^x \sec e^x\,dx$ *Hint:* See Problem 56.

30. $\int e^x \sec^2(e^x)\,dx$

31. $\int \dfrac{\sec^3 x + e^{\sin x}}{\sec x}\,dx$

32. $\int \dfrac{(6t - 1)\sin\sqrt{3t^2 - t - 1}}{\sqrt{3t^2 - t - 1}}\,dt$

33. $\int \dfrac{t^2\cos(t^3 - 2)}{\sin^2(t^3 - 2)}\,dt$

34. $\int \dfrac{1 + \cos 2x}{\sin^2 2x}\,dx$

35. $\int \dfrac{t^2\cos^2(t^3 - 2)}{\sin^2(t^3 - 2)}\,dt$

36. $\int \dfrac{\csc^2 2t}{\sqrt{1 + \cot 2t}}\,dt$

37. $\int \dfrac{e^{\tan^{-1} 2t}}{1 + 4t^2}\,dt$

38. $\int (t + 1)e^{-t^2 - 2t - 5}\,dt$

39. $\int \dfrac{y}{\sqrt{16 - 9y^4}}\,dy$

40. $\int \cosh 3x\,dx$

41. $\int x^2 \sinh x^3\,dx$

42. $\int \dfrac{5}{\sqrt{9 - 4x^2}}\,dx$

43. $\int \dfrac{e^{3t}}{\sqrt{4 - e^{6t}}}\,dt$

44. $\int \dfrac{dt}{2t\sqrt{4t^2 - 1}}$

45. $\int_0^{\pi/2} \dfrac{\sin x}{16 + \cos^2 x}\,dx$

46. $\int_0^1 \dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

47. $\int \dfrac{1}{x^2 + 2x + 5}\,dx$

48. $\int \dfrac{1}{x^2 - 4x + 9}\,dx$

49. $\int \dfrac{dx}{9x^2 + 18x + 10}$

50. $\int \dfrac{dx}{\sqrt{16 + 6x - x^2}}$

51. $\int \dfrac{x + 1}{9x^2 + 18x + 10}\,dx$

52. $\int \dfrac{3 - x}{\sqrt{16 + 6x - x^2}}\,dx$

53. $\int \dfrac{dt}{t\sqrt{2t^2 - 9}}$

54. $\int \dfrac{\tan x}{\sqrt{\sec^2 x - 4}}\,dx$

55. Find the length of the curve $y = \ln(\cos x)$ between $x = 0$ and $x = \pi/4$.

56. Establish the identity

$$\sec x = \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x}$$

and then use it to derive the formula

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

57. Evaluate $\int_0^{2\pi} \dfrac{x|\sin x|}{1 + \cos^2 x}\,dx$. *Hint:* Make the substitution $u = x - \pi$ in the definite integral and then use symmetry properties.

58. Let R be the region bounded by $y = \sin x$ and $y = \cos x$ between $x = -\pi/4$ and $x = 3\pi/4$. Find the volume of the solid obtained when R is revolved about $x = -\pi/4$. *Hint:* Use cylindrical shells to write a single integral, make the substitution $u = x - \pi/4$, and apply symmetry properties.

Answers to Concepts Review: **1.** elementary function **2.** $\int u^5\,du$ **3.** e^x **4.** $\int_1^2 u^3\,du$

7.2 Integration by Parts

If integration by substitution fails, it may be possible to use a double substitution, better known as *integration by parts*. This method is based on the integration of the formula for the derivative of a product of two functions.

Let $u = u(x)$ and $v = v(x)$. Then

$$D_x[u(x)v(x)] = u(x)v'(x) + v(x)u'(x)$$

or

$$u(x)v'(x) = D_x[u(x)v(x)] - v(x)u'(x)$$

By integrating both sides of this equation, we obtain

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

Since $dv = v'(x)\,dx$ and $du = u'(x)\,dx$, the preceding equation is usually written symbolically as follows:

Integration by Parts: Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

The corresponding formula for definite integrals is

$$\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx$$

Figure 1 illustrates a geometric interpretation of integration by parts for definite integrals. We abbreviate this as follows:

Integration by Parts: Definite Integrals

$$\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du$$

These formulas allow us to shift the problem of integrating $u\,dv$ to that of integrating $v\,du$. Success depends on the proper choice of u and dv, which comes with practice.

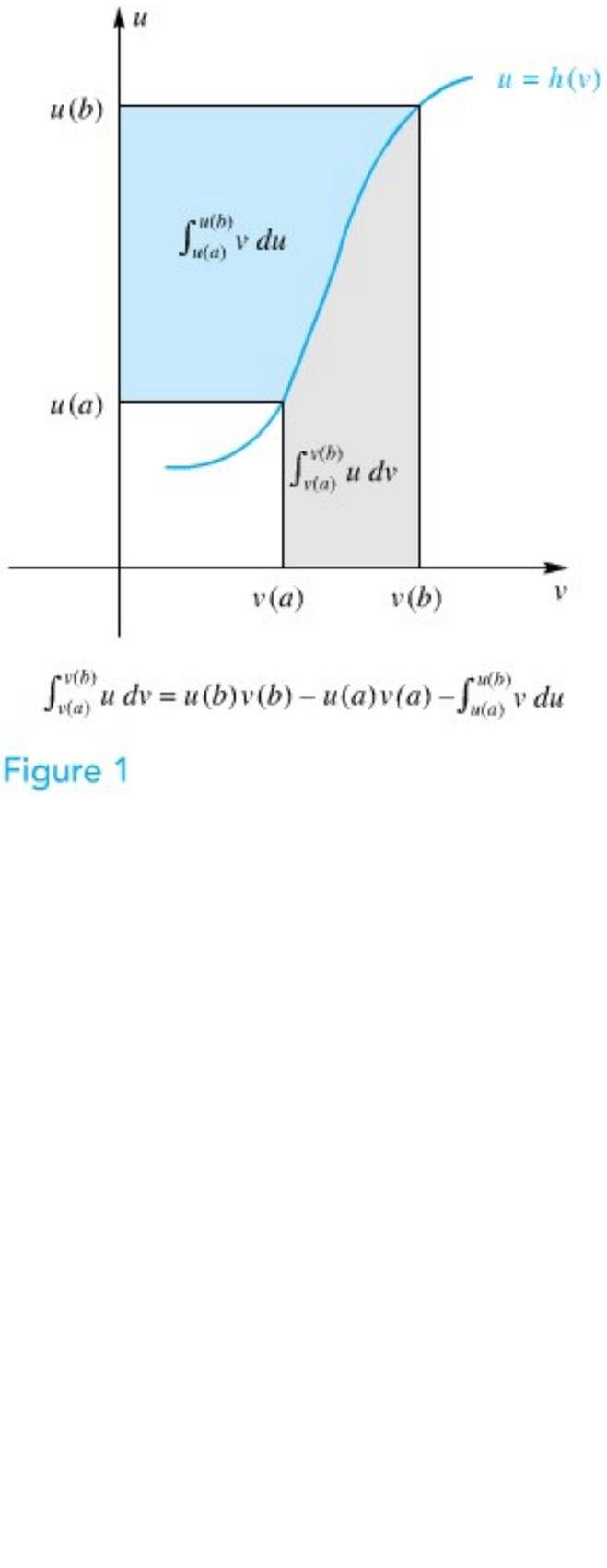


Figure 1

EXAMPLE 1 Find $\int x \cos x\,dx$.

SOLUTION We wish to write $x \cos x\,dx$ as $u\,dv$. One possibility is to let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ and $v = \int \cos x\,dx = \sin x$ (we can omit the arbitrary constant at this stage). Here is a summary of this double substitution in a convenient format.

$$u = x \qquad dv = \cos x\,dx$$
$$du = dx \qquad v = \sin x$$

The formula for integration by parts gives

$$\int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\,\underbrace{dx}_{du}$$
$$= x \sin x + \cos x + C$$

We were successful on our first try. Another substitution would be

$$u = \cos x \qquad dv = x\,dx$$

$$du = -\sin x\,dx \qquad v = \frac{x^2}{2}$$

This time the formula for integration by parts gives

$$\int \underbrace{(\cos x)}_{u}\,\underbrace{x\,dx}_{dv} = \underbrace{(\cos x)}_{u}\,\underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v}\,\underbrace{(-\sin x\,dx)}_{du}$$

which is correct but not helpful. The new integral on the right-hand side is more complicated than the original one. Thus, we see the importance of a wise choice for u and dv.

EXAMPLE 2 Find $\int_1^2 \ln x\,dx$.

SOLUTION We make the following substitutions:

$$u = \ln x \qquad dv = dx$$

$$du = \left(\frac{1}{x}\right)dx \qquad v = x$$

Then

$$\int_1^2 \ln x\,dx = [x \ln x]_1^2 - \int_1^2 x\frac{1}{x}dx$$

$$= 2 \ln 2 - \int_1^2 dx$$

$$= 2 \ln 2 - 1 \approx 0.386$$

EXAMPLE 3 Find $\int \arcsin x\,dx$.

SOLUTION We make the substitutions

$$u = \arcsin x \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1 - x^2}}dx \qquad v = x$$

Then

$$\int \arcsin x\,dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}}\,dx$$

$$= x \arcsin x + \frac{1}{2}\int (1 - x^2)^{-1/2}(-2x\,dx)$$

$$= x \arcsin x + \frac{1}{2}\cdot 2(1 - x^2)^{1/2} + C$$

$$= x \arcsin x + \sqrt{1 - x^2} + C$$

EXAMPLE 4 Find $\int_1^2 t^6 \ln t\,dt$.

SOLUTION We make the following substitutions

$$u = \ln t \qquad dv = t^6\,dt$$
$$du = \frac{1}{t}dt \qquad v = \frac{1}{7}t^7$$

Then

$$\int_1^2 t^6 \ln t\,dt = \left[\frac{1}{7}t^7 \ln t\right]_1^2 - \int_1^2 \frac{1}{7}t^7\left(\frac{1}{t}dt\right)$$
$$= \frac{1}{7}(128 \ln 2 - \ln 1) - \frac{1}{7}\int_1^2 t^6\,dt$$
$$= \frac{128}{7}\ln 2 - \frac{1}{49}[t^7]_1^2$$
$$= \frac{128}{7}\ln 2 - \frac{127}{49} \approx 10.083$$

■

Repeated Integration by Parts Sometimes it is necessary to apply integration by parts several times.

EXAMPLE 5 Find $\int x^2 \sin x\,dx$.

SOLUTION Let

$$u = x^2 \qquad dv = \sin x\,dx$$
$$du = 2x\,dx \qquad v = -\cos x$$

Then

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2\int x \cos x\,dx$$

We have improved our situation (the exponent on x has gone from 2 to 1), which suggests reapplying integration by parts to the integral on the right. Actually, we did this integration in Example 1, so we will make use of the result obtained there.

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2(x \sin x + \cos x + C)$$
$$= -x^2 \cos x + 2x \sin x + 2 \cos x + K$$

■

EXAMPLE 6 Find $\int e^x \sin x\,dx$.

SOLUTION Take $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. Thus,

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

which does not seem to have improved things—but does not leave us any worse off either. So, let's not give up and try integration by parts again. In the integral on the right, let $u = e^x$ and $dv = \cos x\,dx$, so $du = e^x\,dx$ and $v = \sin x$. Then

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

When we substitute this in our first result, we get

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

By moving the last term to the left side and combining terms, we obtain

$$2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + C$$

from which

$$\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + K$$ ■

The fact that the integral we wanted to find reappeared on the right side is what made Example 6 work.

Reduction Formulas A formula of the form

$$\int f^n(x)g(x)\, dx = h(x) + \int f^k(x)\, g(x)\, dx$$

where $k < n$, is called a **reduction formula** (the exponent on f is reduced). Such formulas can often be obtained via integration by parts.

EXAMPLE 7 Derive a reduction formula for $\int \sin^n x\, dx$.

SOLUTION Let $u = \sin^{n-1} x$ and $dv = \sin x\, dx$. Then

$$du = (n-1)\sin^{n-2} x \cos x\, dx \quad \text{and} \quad v = -\cos x$$

from which

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

If we replace $\cos^2 x$ by $1 - \sin^2 x$ in the last integral, we obtain

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

After combining the first and last integrals above and solving for $\int \sin^n x\, dx$, we get the reduction formula (valid for $n \geq 2$)

$$\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$ ■

EXAMPLE 8 Use the reduction formula above to evaluate $\int_0^{\pi/2} \sin^8 x\, dx$.

SOLUTION Note first that

$$\int_0^{\pi/2} \sin^n x\, dx = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

Thus,

$$\int_0^{\pi/2} \sin^8 x\,dx = \frac{7}{8}\int_0^{\pi/2} \sin^6 x\,dx$$
$$= \frac{7}{8}\cdot\frac{5}{6}\int_0^{\pi/2} \sin^4 x\,dx$$
$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\int_0^{\pi/2} \sin^2 x\,dx$$
$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} 1\,dx$$
$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35}{256}\pi$$

The general formula for $\int_0^{\pi/2} \sin^n x\,dx$ can be found in a similar way (Formula 113 at the back of the book).

Concepts Review

1. The integration-by-parts formula says that $\int u\,dv =$ _____.

2. To apply this formula to $\int x \sin x\,dx$, let $u =$ _____ and $dv =$ _____.

3. Applying the integration-by-parts formula yields the value _____ for $\int_0^{\pi/2} x \sin x\,dx$.

4. A formula that expresses $\int f^n(x)\,g(x)\,dx$ in terms of $\int f^k(x)\,g(x)\,dx$, where $k < n$, is called a _____ formula.

Problem Set 7.2

In Problems 1–36, use integration by parts to evaluate each integral.

1. $\int xe^x\,dx$

2. $\int xe^{3x}\,dx$

3. $\int te^{5t+\pi}\,dt$

4. $\int (t+7)e^{2t+3}\,dt$

5. $\int x\cos x\,dx$

6. $\int x\sin 2x\,dx$

7. $\int (t-3)\cos(t-3)\,dt$

8. $\int (x-\pi)\sin x\,dx$

9. $\int t\sqrt{t+1}\,dt$

10. $\int t\sqrt[3]{2t+7}\,dt$

11. $\int \ln 3x\,dx$

12. $\int \ln(7x^5)\,dx$

13. $\int \arctan x\,dx$

14. $\int \arctan 5x\,dx$

15. $\int \frac{\ln x}{x^2}\,dx$

16. $\int_2^3 \frac{\ln 2x^5}{x^2}\,dx$

17. $\int_1^e \sqrt{t}\ln t\,dt$

18. $\int_1^5 \sqrt{2x}\ln x^3\,dx$

19. $\int z^3\ln z\,dz$

20. $\int t\arctan t\,dt$

21. $\int \arctan(1/t)\,dt$

22. $\int t^5\ln(t^7)\,dt$

23. $\int_{\pi/6}^{\pi/2} x\csc^2 x\,dx$

24. $\int_{\pi/6}^{\pi/4} x\sec^2 x\,dx$

25. $\int x^5\sqrt{x^3+4}\,dx$

26. $\int x^{13}\sqrt{x^7+1}\,dx$

27. $\int \frac{t^7}{(7-3t^4)^{3/2}}\,dt$

28. $\int x^3\sqrt{4-x^2}\,dx$

29. $\int \frac{z^7}{(4-z^4)^2}\,dz$

30. $\int x\cosh x\,dx$

31. $\int x\sinh x\,dx$

32. $\int \frac{\ln x}{\sqrt{x}}\,dx$

33. $\int x(3x+10)^{49}\,dx$

34. $\int_0^1 t(t-1)^{12}\,dt$

35. $\int x\,2^x\,dx$

36. $\int z\,a^z\,dz$

In Problems 37–48, apply integration by parts twice to evaluate each integral (see Examples 5 and 6).

37. $\int x^2e^x\,dx$

38. $\int x^5e^{x^2}\,dx$

39. $\int \ln^2 z\,dz$

40. $\int \ln^2 x^{20}\,dx$

41. $\int e^t\cos t\,dt$

42. $\int e^{at}\sin t\,dt$

43. $\int x^2\cos x\,dx$

44. $\int r^2\sin r\,dr$

45. $\int \sin(\ln x)\,dx$

46. $\int \cos(\ln x)\,dx$

47. $\int (\ln x)^3\,dx$ *Hint:* Use Problem 39.

48. $\int (\ln x)^4\,dx$ *Hint:* Use Problems 39 and 47.

In Problems 49–54, use integration by parts to derive the given formula.

49. $\int \sin x \sin 3x\,dx = -\frac{3}{8}\sin x \cos 3x + \frac{1}{8}\cos x \sin 3x + C$

50. $\int \cos 5x \sin 7x\,dx = -\frac{7}{24}\cos 5x \cos 7x - \frac{5}{24}\sin 5x \sin 7x + C$

51. $\int e^{\alpha z}\sin \beta z\,dz = \dfrac{e^{\alpha z}(\alpha \sin \beta z - \beta \cos \beta z)}{\alpha^2 + \beta^2} + C$

52. $\int e^{\alpha z}\cos \beta z\,dz = \dfrac{e^{\alpha z}(\alpha \cos \beta z + \beta \sin \beta z)}{\alpha^2 + \beta^2} + C$

53. $\int x^\alpha \ln x\,dx = \dfrac{x^{\alpha+1}}{\alpha+1}\ln x - \dfrac{x^{\alpha+1}}{(\alpha+1)^2} + C, \alpha \neq -1$

54. $\int x^\alpha (\ln x)^2\,dx = \dfrac{x^{\alpha+1}}{\alpha+1}(\ln x)^2 - 2\dfrac{x^{\alpha+1}}{(\alpha+1)^2}\ln x + 2\dfrac{x^{\alpha+1}}{(\alpha+1)^3} + C, \alpha \neq -1$

In Problems 55–61, derive the given reduction formula using integration by parts.

55. $\int x^\alpha e^{\beta x}\,dx = \dfrac{x^\alpha e^{\beta x}}{\beta} - \dfrac{\alpha}{\beta}\int x^{\alpha-1}e^{\beta x}\,dx$

56. $\int x^\alpha \sin \beta x\,dx = -\dfrac{x^\alpha \cos \beta x}{\beta} + \dfrac{\alpha}{\beta}\int x^{\alpha-1}\cos \beta x\,dx$

57. $\int x^\alpha \cos \beta x\,dx = \dfrac{x^\alpha \sin \beta x}{\beta} - \dfrac{\alpha}{\beta}\int x^{\alpha-1}\sin \beta x\,dx$

58. $\int (\ln x)^\alpha\,dx = x(\ln x)^\alpha - \alpha\int (\ln x)^{\alpha-1}\,dx$

59. $\int (a^2 - x^2)^\alpha\,dx = x(a^2 - x^2)^\alpha + 2\alpha\int x^2(a^2 - x^2)^{\alpha-1}\,dx$

60. $\int \cos^\alpha x\,dx = \dfrac{\cos^{\alpha-1} x \sin x}{\alpha} + \dfrac{\alpha-1}{\alpha}\int \cos^{\alpha-2} x\,dx$

61. $\int \cos^\alpha \beta x\,dx = \dfrac{\cos^{\alpha-1}\beta x \sin \beta x}{\alpha\beta} + \dfrac{\alpha-1}{\alpha}\int \cos^{\alpha-2}\beta x\,dx$

62. Use Problem 55 to derive

$$\int x^4 e^{3x}\,dx = \tfrac{1}{3}x^4e^{3x} - \tfrac{4}{9}x^3e^{3x} + \tfrac{4}{9}x^2e^{3x} - \tfrac{8}{27}xe^{3x} + \tfrac{8}{81}e^{3x} + C$$

63. Use Problems 56 and 57 to derive

$$\int x^4 \cos 3x\,dx = \tfrac{1}{3}x^4 \sin 3x + \tfrac{4}{9}x^3 \cos 3x - \tfrac{4}{9}x^2 \sin 3x - \tfrac{8}{27}x \cos 3x + \tfrac{8}{81}\sin 3x + C.$$

64. Use Problem 61 to derive

$$\int \cos^6 3x\,dx = \tfrac{1}{18}\sin 3x \cos^5 3x + \tfrac{5}{72}\sin 3x \cos^3 3x + \tfrac{5}{48}\sin 3x \cos 3x + \tfrac{5}{16}x + C.$$

≈ **65.** Find the area of the region bounded by the curve $y = \ln x$, the x-axis, and the line $x = e$.

≈ **66.** Find the volume of the solid generated by revolving the region of Problem 65 about the x-axis.

≈ **67.** Find the area of the region bounded by the curves $y = 3e^{-x/3}$, $y = 0$, $x = 0$, and $x = 9$. Make a sketch.

≈ **68.** Find the volume of the solid generated by revolving the region described in Problem 67 about the x-axis.

≈ **69.** Find the area of the region bounded by the graphs of $y = x \sin x$ and $y = x \cos x$ from $x = 0$ to $x = \pi/4$.

≈ **70.** Find the volume of the solid obtained by revolving the region under the graph of $y = \sin(x/2)$ from $x = 0$ to $x = 2\pi$ about the y-axis.

≈ **71.** Find the centroid (see Section 5.6) of the region bounded by $y = \ln x^2$ and the x-axis from $x = 1$ to $x = e$.

72. Evaluate the integral $\int \cot x \csc^2 x\,dx$ by parts in two different ways:

(a) By differentiating $\cot x$ (b) By differentiating $\csc x$

(c) Show that the two results are equivalent up to a constant.

73. If $p(x)$ is a polynomial of degree n and $G_1, G_2, \ldots, G_{n+1}$, are successive antiderivatives of a function g, then, by repeated integration by parts,

$$\int p(x)g(x)\,dx = p(x)G_1(x) - p'(x)G_2(x) + p''(x)G_3(x) - \cdots + (-1)^n p^{(n)}(x)G_{n+1}(x) + C$$

Use this result to find each of the following:

(a) $\int (x^3 - 2x)e^x\,dx$ (b) $\int (x^2 - 3x + 1)\sin x\,dx$

≈ **74.** The graph of $y = x \sin x$ for $x \geq 0$ is sketched in Figure 2.

(a) Find a formula for the area of the nth arch.

(b) The second arch is revolved about the y-axis. Find the volume of the resulting solid.

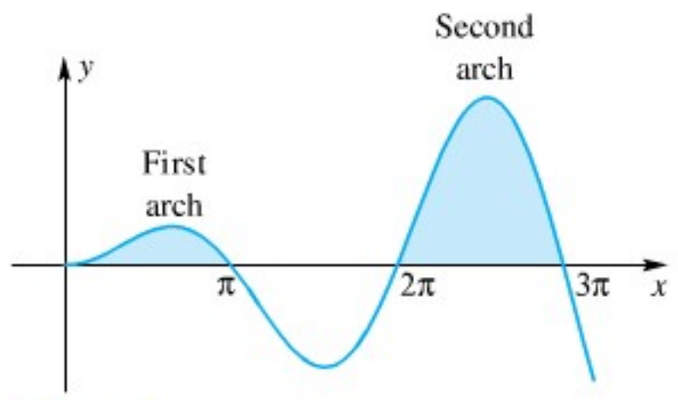


Figure 2

75. The quantity $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ plays an important role in applied mathematics. Show that if $f'(x)$ is continuous on $[-\pi, \pi]$, then $\lim\limits_{n\to\infty} a_n = 0$. *Hint:* Integration by parts.

76. Let $G_n = \sqrt[n]{(n+1)(n+2)\cdots(n+n)}$. Show that $\lim\limits_{n\to\infty}(G_n/n) = 4/e$. *Hint:* Consider $\ln(G_n/n)$, recognize it as a Riemann sum, and use Example 2.

77. Find the error in the following "proof" that $0 = 1$. In $\int (1/t)\,dt$, set $u = 1/t$ and $dv = dt$. Then $du = -t^{-2}\,dt$ and $uv = 1$. Integration by parts gives

$$\int (1/t)\,dt = 1 - \int (-1/t)\,dt$$

or $0 = 1$.

78. Suppose that you want to evaluate the integral

$$\int e^{5x}(4\cos 7x + 6\sin 7x)\,dx$$

and you know from experience that the result will be of the form $e^{5x}(C_1\cos 7x + C_2\sin 7x) + C_3$. Compute C_1 and C_2 by differentiating the result and setting it equal to the integrand.

Many surprising theoretical results can be derived through the use of integration by parts. In all cases, one starts with an integral. We explore two of these results here.

79. Show that

$$\int_a^b f(x)\,dx = [xf(x)]_a^b - \int_a^b xf'(x)\,dx$$
$$= [(x-a)f(x)]_a^b - \int_a^b (x-a)f'(x)\,dx$$

80. Using Problem 79 and replacing f by f', show that

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$
$$= f'(b)(b-a) - \int_a^b (x-a)f''(x)\,dx$$
$$= f'(a)(b-a) - \int_a^b (x-b)f''(x)\,dx$$

81. Show that

$$f(t) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i + \int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)\,dx,$$

provided that f can be differentiated $n+1$ times.

82. The *Beta function*, which is important in many branches of mathematics, is defined as

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx,$$

with the condition that $\alpha \geq 1$ and $\beta \geq 1$.

(a) Show by a change of variables that

$$B(\alpha,\beta) = \int_0^1 x^{\beta-1}(1-x)^{\alpha-1}\,dx = B(\beta,\alpha)$$

(b) Integrate by parts to show that

$$B(\alpha,\beta) = \frac{\alpha-1}{\beta}B(\alpha-1,\beta+1) = \frac{\beta-1}{\alpha}B(\alpha+1,\beta-1)$$

(c) Assume now that $\alpha = n$ and $\beta = m$, and that n and m are positive integers. By using the result in part (b) repeatedly, show that

$$B(n,m) = \frac{(n-1)!\,(m-1)!}{(n+m-1)!}$$

This result is valid even in the case where n and m are not integers, provided that we can give meaning to $(n-1)!$, $(m-1)!$, and $(n+m-1)!$.

83. Suppose that $f(t)$ has the property that $f'(a) = f'(b) = 0$ and that $f(t)$ has two continuous derivatives. Use integration by parts to prove that $\int_a^b f''(t)f(t)\,dt \leq 0$. *Hint:* Use integration by parts by differentiating $f(t)$ and integrating $f''(t)$. This result has many applications in the field of applied mathematics.

84. Derive the formula

$$\int_0^x \left(\int_0^t f(z)\,dz\right) dt = \int_0^x f(t)(x-t)\,dt$$

using integration by parts.

85. Generalize the formula given in Problem 84 to one for an n-fold iterated integral

$$\int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_1 =$$
$$\frac{1}{(n-1)!}\int_0^x f(t_1)(x-t_1)^{n-1}\,dt_1$$

86. If $P_n(x)$ is a polynomial of degree n, show that

$$\int e^x P_n(x)\,dx = e^x \sum_{j=0}^{n} (-1)^j \frac{d^j P_n(x)}{dx^j}$$

87. Use the result from Problem 86 to evaluate

$$\int (3x^4 + 2x^2)e^x\,dx$$

Answers to Concepts Review: **1.** $uv - \int v\,du$ **2.** x; $\sin x\,dx$ **3.** 1 **4.** reduction

7.3 Some Trigonometric Integrals

When we combine the method of substitution with a clever use of trigonometric identities, we can integrate a wide variety of trigonometric forms. We consider five commonly encountered types.

1. $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$
2. $\int \sin^m x \cos^n x\,dx$
3. $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$, $\int \cos mx \cos nx\,dx$
4. $\int \tan^n x\,dx$, $\int \cot^n x\,dx$
5. $\int \tan^m x \sec^n x\,dx$, $\int \cot^m x \csc^n x\,dx$

Useful Identities

Some trigonometric identities needed in this section are the following.

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Half-Angle Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Type 1 $\left(\int \sin^n x\,dx,\ \int \cos^n x\,dx\right)$ Consider first the case where n is an odd positive integer. After taking out either the factor $\sin x$ or $\cos x$, use the identity $\sin^2 x + \cos^2 x = 1$.

EXAMPLE 1 **(n Odd)** Find $\int \sin^5 x\,dx$.

SOLUTION

$$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx$$

$$= \int (1 - \cos^2 x)^2 \sin x\,dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x) \sin x\,dx$$

$$= -\int (1 - 2\cos^2 x + \cos^4 x)(-\sin x\,dx)$$

$$= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C$$

■

EXAMPLE 2 **(n Even)** Find $\int \sin^2 x\,dx$ and $\int \cos^4 x\,dx$.

SOLUTION Here we make use of half-angle identities.

$$\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx$$

$$= \frac{1}{2}\int dx - \frac{1}{4}\int (\cos 2x)(2\,dx)$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$\int \cos^4 x\,dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int dx + \frac{1}{4}\int (\cos 2x)(2)\,dx + \frac{1}{8}\int (1 + \cos 4x)\,dx$$

$$= \frac{3}{8}\int dx + \frac{1}{4}\int \cos 2x(2\,dx) + \frac{1}{32}\int \cos 4x(4\,dx)$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

■

Type 2 $\left(\int \sin^m x \cos^n x\,dx\right)$ If either m or n is an odd positive integer and the other exponent is any number, we factor out $\sin x$ or $\cos x$ and use the identity $\sin^2 x + \cos^2 x = 1$.

EXAMPLE 3 **(m or n Odd)** Find $\int \sin^3 x \cos^{-4} x\,dx$.

SOLUTION

$$\begin{aligned}\int \sin^3 x \cos^{-4} x\, dx &= \int (1 - \cos^2 x)(\cos^{-4} x)(\sin x)\, dx \\ &= -\int (\cos^{-4} x - \cos^{-2} x)(-\sin x\, dx) \\ &= -\left[\frac{(\cos x)^{-3}}{-3} - \frac{(\cos x)^{-1}}{-1}\right] + C \\ &= \frac{1}{3}\sec^3 x - \sec x + C\end{aligned}$$

■

If both m and n are even positive integers, we use half-angle identities to reduce the degree of the integrand. Example 4 gives an illustration.

EXAMPLE 4 **(Both m and n Even)** Find $\int \sin^2 x \cos^4 x\, dx$.

SOLUTION

$$\int \sin^2 x \cos^4 x\, dx$$

$$\begin{aligned}&= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\ &= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\ &= \frac{1}{8}\int \left[1 + \cos 2x - \frac{1}{2}(1 + \cos 4x) - (1 - \sin^2 2x)\cos 2x\right] dx \\ &= \frac{1}{8}\int \left[\frac{1}{2} - \frac{1}{2}\cos 4x + \sin^2 2x \cos 2x\right] dx \\ &= \frac{1}{8}\left[\int \frac{1}{2} dx - \frac{1}{8}\int \cos 4x(4\, dx) + \frac{1}{2}\int \sin^2 2x(2\cos 2x\, dx)\right] \\ &= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right] + C\end{aligned}$$

■

Are They Different?

Indefinite integrations may lead to different looking answers. By one method,

$$\begin{aligned}\int \sin x \cos x\, dx &= -\int \cos x(-\sin x)\, dx \\ &= -\tfrac{1}{2}\cos^2 x + C\end{aligned}$$

By a second method,

$$\begin{aligned}\int \sin x \cos x\, dx &= \int \sin x(\cos x)\, dx \\ &= \tfrac{1}{2}\sin^2 x + C\end{aligned}$$

But two such answers should differ by at most a constant. Note, however, that

$$\begin{aligned}\tfrac{1}{2}\sin^2 x + C &= \tfrac{1}{2}(1 - \cos^2 x) + C \\ &= -\tfrac{1}{2}\cos^2 x + \left(\tfrac{1}{2} + C\right)\end{aligned}$$

Now reconcile these answers with a third answer.

$$\begin{aligned}\int \sin x \cos x\, dx &= \tfrac{1}{2}\int \sin 2x\, dx \\ &= -\tfrac{1}{4}\cos 2x + C\end{aligned}$$

Type 3 $\left(\int \sin mx \cos nx\, dx,\ \int \sin mx \sin nx\, dx,\ \int \cos mx \cos nx\, dx\right)$

Integrals of this type occur in many physics and engineering applications. To handle these integrals, we use the product identities.

1. $\sin mx \cos nx = \frac{1}{2}[\sin(m + n)x + \sin(m - n)x]$
2. $\sin mx \sin nx = -\frac{1}{2}[\cos(m + n)x - \cos(m - n)x]$
3. $\cos mx \cos nx = \frac{1}{2}[\cos(m + n)x + \cos(m - n)x]$

EXAMPLE 5 Find $\int \sin 2x \cos 3x\, dx$.

SOLUTION Apply product identity 1.

$$\int \sin 2x \cos 3x\, dx = \frac{1}{2}\int [\sin 5x + \sin(-x)]\, dx$$
$$= \frac{1}{10}\int \sin 5x(5\, dx) - \frac{1}{2}\int \sin x\, dx$$
$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

EXAMPLE 6 If m and n are positive integers, show that

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

SOLUTION Apply product identity 2. If $m \neq n$, then

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = -\frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x - \cos(m-n)x]\, dx$$
$$= -\frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x - \frac{1}{m-n}\sin(m-n)x\right]_{-\pi}^{\pi}$$
$$= 0$$

If $m = n$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = -\frac{1}{2}\int_{-\pi}^{\pi} [\cos 2mx - 1]\, dx$$
$$= -\frac{1}{2}\left[\frac{1}{2m}\sin 2mx - x\right]_{-\pi}^{\pi}$$
$$= -\frac{1}{2}[-2\pi] = \pi$$

EXAMPLE 7 If m and n are positive integers, find

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx$$

SOLUTION Let $u = \pi x/L$, $du = \pi dx/L$. If $x = -L$, then $u = -\pi$, and if $x = L$, then $u = \pi$. Thus,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \frac{L}{\pi}\int_{-\pi}^{\pi} \sin mu \sin nu\, du$$
$$= \begin{cases} \dfrac{L}{\pi}\cdot 0 & \text{if } m \neq n \\ \dfrac{L}{\pi}\cdot \pi & \text{if } m = n \end{cases}$$
$$= \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

Here we have used the result of Example 6.

A number of times in this book we have suggested that you should view things from both an algebraic and a geometric point of view. So far, this section has been entirely algebraic, but with definite integrals such as those in Examples 6 and 7, we have an opportunity to view things geometrically.

Figure 1 shows graphs of $y = \sin 3x \sin 2x$ and $y = \sin(3\pi x/10)\sin(2\pi x/10)$. The graphs suggest that the areas above and below the x-axis are the same, leaving $A_{\text{up}} - A_{\text{down}} = 0$. Examples 6 and 7 confirm this.

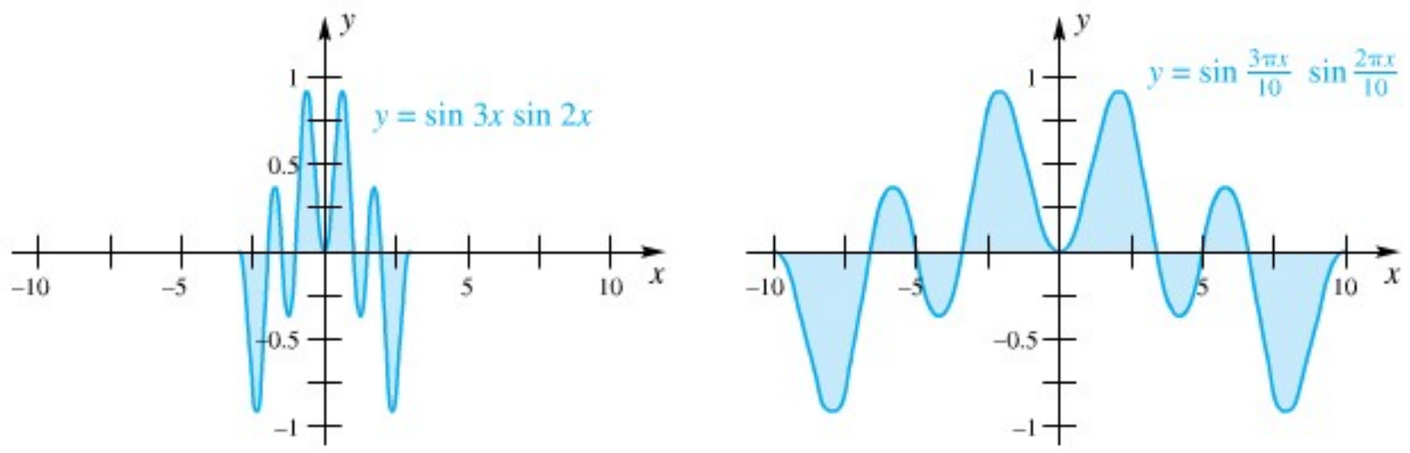


Figure 1

Figure 2 shows graphs of $y = \sin 2x \sin 2x = \sin^2 2x$, $-\pi \le x \le \pi$, and $y = \sin(2\pi x/10)\sin(2\pi x/10) = \sin^2(2\pi x/10)$, $-10 \le x \le 10$. These two graphs look the same, except the one on the right has been stretched horizontally by the factor $10/\pi$. Does it then make sense that the area will increase by this same factor? That would make the shaded area in the figure on the right equal to $10/\pi$ times the shaded area in the figure on the left; that is, the area on the right should be $(10/\pi)\cdot\pi = 10$, which corresponds to the result of Example 7 with $L = 10$.

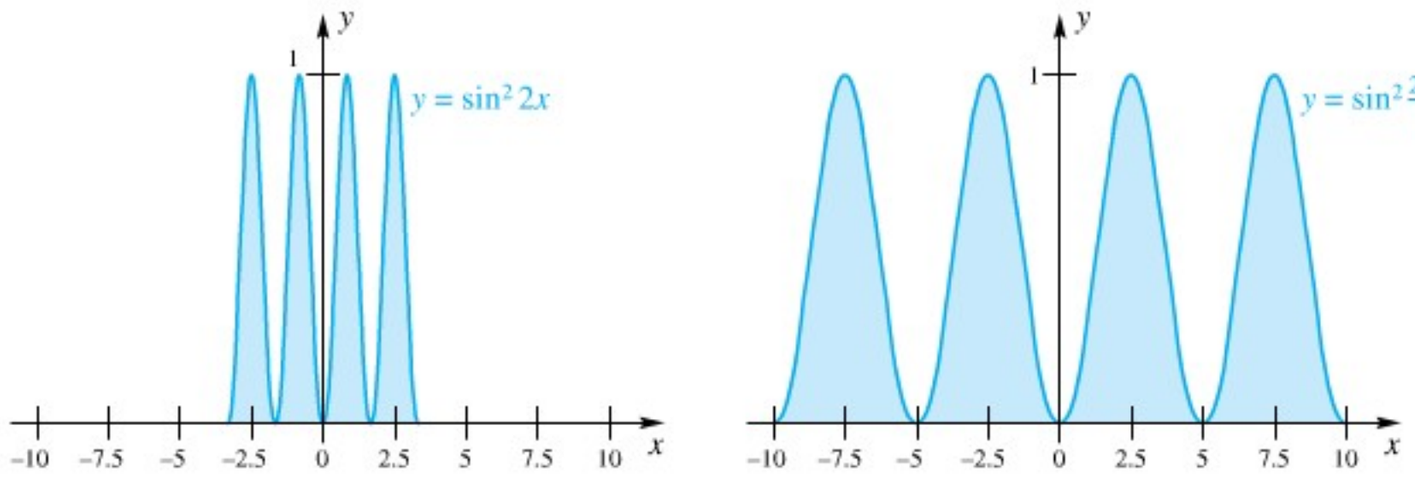


Figure 2

Type 4 $\left(\int \tan^n x\,dx,\ \int \cot^n x\,dx\right)$ In the tangent case, factor out $\tan^2 x = \sec^2 x - 1$; in the cotangent case, factor out $\cot^2 x = \csc^2 x - 1$.

EXAMPLE 8 Find $\int \cot^4 x\,dx$.

SOLUTION

$$\begin{aligned}\int \cot^4 x\,dx &= \int \cot^2 x\,(\csc^2 x - 1)\,dx \\ &= \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx \\ &= -\int \cot^2 x\,(-\csc^2 x\,dx) - \int (\csc^2 x - 1)\,dx \\ &= -\tfrac{1}{3}\cot^3 x + \cot x + x + C\end{aligned}$$

■

EXAMPLE 9 Find $\int \tan^5 x\,dx$.

SOLUTION

$$\begin{aligned}\int \tan^5 x\,dx &= \int \tan^3 x\,(\sec^2 x - 1)\,dx \\ &= \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx \\ &= \int \tan^3 x\,(\sec^2 x\,dx) - \int \tan x\,(\sec^2 x - 1)\,dx \\ &= \int \tan^3 x\,(\sec^2 x\,dx) - \int \tan x\,(\sec^2 x\,dx) + \int \tan x\,dx \\ &= \tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x - \ln|\cos x| + C\end{aligned}$$

■

Type 5 $\left(\int \tan^m x \sec^n x\,dx,\ \int \cot^m x \csc^n x\,dx\right)$

EXAMPLE 10 **(*n* Even, *m* Any Number)** Find $\int \tan^{-3/2} x \sec^4 x\,dx$.

SOLUTION

$$\begin{aligned}\int \tan^{-3/2} x \sec^4 x\,dx &= \int (\tan^{-3/2} x)(1 + \tan^2 x)\sec^2 x\,dx \\ &= \int (\tan^{-3/2} x)\sec^2 x\,dx + \int (\tan^{1/2} x)\sec^2 x\,dx \\ &= -2\tan^{-1/2} x + \tfrac{2}{3}\tan^{3/2} x + C\end{aligned}$$

■

EXAMPLE 11 **(*m* Odd, *n* Any Number)** Find $\int \tan^3 x \sec^{-1/2} x\,dx$.

SOLUTION

$$\begin{aligned}\int \tan^3 x \sec^{-1/2} x\,dx &= \int (\tan^2 x)(\sec^{-3/2} x)(\sec x \tan x)\,dx \\ &= \int (\sec^2 x - 1)\sec^{-3/2} x\,(\sec x \tan x\,dx) \\ &= \int \sec^{1/2} x\,(\sec x \tan x\,dx) - \int \sec^{-3/2} x\,(\sec x \tan x\,dx) \\ &= \tfrac{2}{3}\sec^{3/2} x + 2\sec^{-1/2} x + C\end{aligned}$$

■

Concepts Review

1. To handle $\int \cos^2 x\,dx$, we first rewrite it as _____.
2. To handle $\int \cos^3 x\,dx$, we first rewrite it as _____.
3. To handle $\int \sin^2 x \cos^3 x\,dx$, we first rewrite it as _____.
4. To handle $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx$, where $m \neq n$, we use the trigonometric identity _____.

Problem Set 7.3

In Problems 1–28, perform the indicated integrations.

1. $\int \sin^2 x\,dx$

2. $\int \sin^4 6x\,dx$

3. $\int \sin^3 x\,dx$

4. $\int \cos^3 x\,dx$

5. $\int_0^{\pi/2} \cos^5 \theta\,d\theta$

6. $\int_0^{\pi/2} \sin^6 \theta\,d\theta$

7. $\int \sin^5 4x \cos^2 4x\,dx$

8. $\int (\sin^3 2t)\sqrt{\cos 2t}\,dt$

9. $\int \cos^3 3\theta \sin^{-2} 3\theta\,d\theta$

10. $\int \sin^{1/2} 2z \cos^3 2z\,dz$

11. $\int \sin^4 3t \cos^4 3t\,dt$

12. $\int \cos^6 \theta \sin^2 \theta\,d\theta$

13. $\int \sin 4y \cos 5y\,dy$

14. $\int \cos y \cos 4y\,dy$

15. $\int \sin^4\left(\frac{w}{2}\right)\cos^2\left(\frac{w}{2}\right)dw$

16. $\int \sin 3t \sin t\,dt$

17. $\int x \cos^2 x \sin x\,dx$ *Hint:* Use integration by parts.

18. $\int x \sin^3 x \cos x\,dx$

19. $\int \tan^4 x\,dx$

20. $\int \cot^4 x\,dx$

21. $\int \tan^3 x\,dx$

22. $\int \cot^3 2t\,dt$

23. $\int \tan^5\left(\frac{\theta}{2}\right)d\theta$

24. $\int \cot^5 2t\,dt$

25. $\int \tan^{-3} x \sec^4 x\,dx$

26. $\int \tan^{-3/2} x \sec^4 x\,dx$

27. $\int \tan^3 x \sec^2 x\,dx$

28. $\int \tan^3 x \sec^{-1/2} x\,dx$

29. Find $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx$, $m \neq n$; m, n integers.

30. Find $\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}dx$, $m \neq n$, m, n integers.

≈ **31.** The region bounded by $y = x + \sin x$, $y = 0$, $x = \pi$, is revolved about the x-axis. Find the volume of the resulting solid.

≈ **32.** The region bounded by $y = \sin^2(x^2)$, $y = 0$, and $x = \sqrt{\pi/2}$ is revolved about the y-axis. Find the volume of the resulting solid.

33. Let $f(x) = \sum_{n=1}^{N} a_n \sin(nx)$. Use Example 6 to show each of the following for a positive integer m.

(a) $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \begin{cases} a_m & \text{if } m \leq N \\ 0 & \text{if } m > N \end{cases}$

(b) $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \sum_{n=1}^{N} a_n^2$

Note: Integrals of this type occur in a subject called *Fourier series*, which has applications to heat, vibrating strings, and other physical phenomena.

34. Show that

$$\lim_{n\to\infty} \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cdots\cos\frac{x}{2^n} = \frac{\sin x}{x}$$

by completing the following steps.

(a) $\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n} =$

$$\left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{1}{2^{n-1}}$$

(See Problem 46 of Section 0.7.)

(b) Recognize a Riemann sum leading to a definite integral.

(c) Evaluate the definite integral.

35. Use the result of Problem 34 to obtain the famous formula of François Viète (1540–1603):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

36. The shaded region (Figure 3) between one arch of $y = \sin x$, $0 \leq x \leq \pi$, and the line $y = k$, $0 \leq k \leq 1$, is revolved about the line $y = k$, generating a solid S. Determine k so that S has

(a) minimum volume and (b) maximum volume.

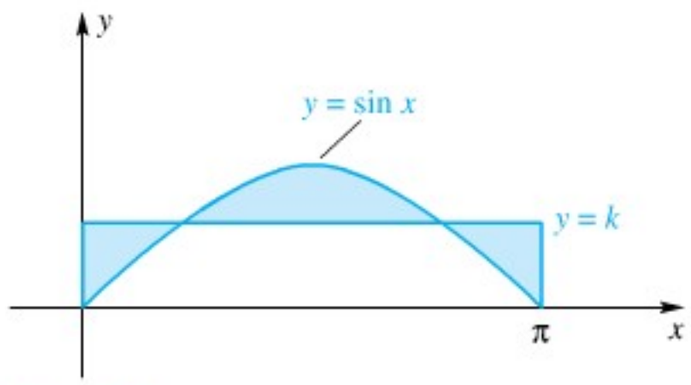


Figure 3

Answers to Concepts Review: **1.** $\int [(1 + \cos 2x)/2]\,dx$ **2.** $\int (1 - \sin^2 x)\cos x\,dx$ **3.** $\int \sin^2 x(1 - \sin^2 x)\cos x\,dx$ **4.** $\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

7.4 Rationalizing Substitutions

Radicals in an integrand are often troublesome and we usually try to get rid of them. Often an appropriate substitution will rationalize the integrand.

Integrands Involving $\sqrt[n]{ax+b}$ If $\sqrt[n]{ax+b}$ appears in an integral, the substitution $u = \sqrt[n]{ax+b}$ will eliminate the radical.

EXAMPLE 1 Find $\displaystyle\int \frac{dx}{x-\sqrt{x}}$.

SOLUTION Let $u = \sqrt{x}$, so $u^2 = x$ and $2u\,du = dx$. Then

$$\int \frac{dx}{x-\sqrt{x}} = \int \frac{2u}{u^2-u}\,du = 2\int \frac{1}{u-1}\,du$$
$$= 2\ln|u-1| + C = 2\ln|\sqrt{x}-1| + C$$

EXAMPLE 2 Find $\displaystyle\int x\sqrt[3]{x-4}\,dx$.

SOLUTION Let $u = \sqrt[3]{x-4}$, so $u^3 = x-4$ and $3u^2\,du = dx$. Then

$$\int x\sqrt[3]{x-4}\,dx = \int (u^3+4)u\cdot(3u^2\,du) = 3\int (u^6+4u^3)\,du$$
$$= 3\left[\frac{u^7}{7}+u^4\right] + C = \frac{3}{7}(x-4)^{7/3} + 3(x-4)^{4/3} + C$$

EXAMPLE 3 Find $\displaystyle\int x\sqrt[5]{(x+1)^2}\,dx$.

SOLUTION Let $u = (x+1)^{1/5}$, so $u^5 = x+1$ and $5u^4\,du = dx$. Then

$$\int x(x+1)^{2/5}\,dx = \int (u^5-1)u^2\cdot 5u^4\,du$$
$$= 5\int (u^{11}-u^6)\,du = \tfrac{5}{12}u^{12} - \tfrac{5}{7}u^7 + C$$
$$= \tfrac{5}{12}(x+1)^{12/5} - \tfrac{5}{7}(x+1)^{7/5} + C$$

Integrands Involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$ To rationalize these three expressions, we may assume that a is positive and make the following trigonometric substitutions.

	Radical	Substitution	Restriction on t
1.	$\sqrt{a^2-x^2}$	$x = a\sin t$	$-\pi/2 \le t \le \pi/2$
2.	$\sqrt{a^2+x^2}$	$x = a\tan t$	$-\pi/2 < t < \pi/2$
3.	$\sqrt{x^2-a^2}$	$x = a\sec t$	$0 \le t \le \pi,\ t \ne \pi/2$

Now note the simplifications that these substitutions achieve.

1. $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 t} = \sqrt{a^2\cos^2 t} = |a\cos t| = a\cos t$
2. $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2 t} = \sqrt{a^2\sec^2 t} = |a\sec t| = a\sec t$
3. $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 t - a^2} = \sqrt{a^2\tan^2 t} = |a\tan t| = \pm a\tan t$

The restrictions on t allowed us to remove the absolute value signs in the first two cases, but they also achieved something else. These restrictions are exactly the ones we introduced in Section 6.7 in order to make sine, tangent, and secant invertible functions. This means that we can solve the substitution equations for t in each case, and this will allow us to write our final answers in the following examples in terms of x.

EXAMPLE 4 Find $\int \sqrt{a^2 - x^2}\, dx$.

SOLUTION We make the substitution

$$x = a \sin t, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

Then $dx = a \cos t\, dt$ and $\sqrt{a^2 - x^2} = a \cos t$. Thus,

$$\int \sqrt{a^2 - x^2}\, dx = \int a \cos t \cdot a \cos t\, dt = a^2 \int \cos^2 t\, dt$$

$$= \frac{a^2}{2} \int (1 + \cos 2t)\, dt$$

$$= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{a^2}{2}(t + \sin t \cos t) + C$$

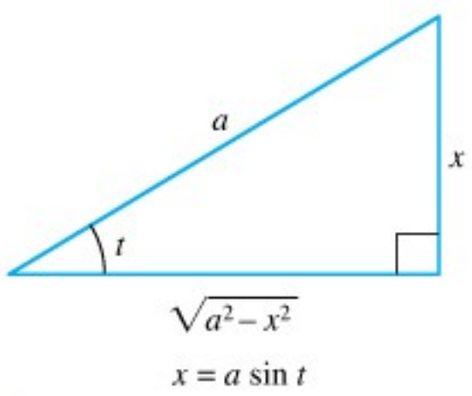


Figure 1

Now, $x = a \sin t$ is equivalent to $x/a = \sin t$ and, since t was restricted to make the sine function invertible,

$$t = \sin^{-1}\left(\frac{x}{a}\right)$$

Using the right triangle in Figure 1 (as we did in Section 6.8), we see that

$$\cos t = \cos\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{1 - \frac{x^2}{a^2}} = \frac{1}{a}\sqrt{a^2 - x^2}$$

Thus,

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$ ■

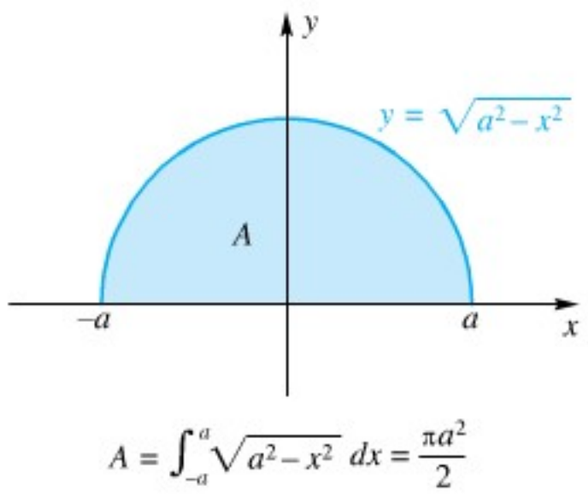


Figure 2

The result in Example 4 allows us to calculate the following definite integral, which represents the area of a semicircle (Figure 2). Thus, calculus confirms a result that we already know.

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \left[\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_{-a}^{a} = \frac{a^2}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi a^2}{2}$$

EXAMPLE 5 Find $\int \frac{dx}{\sqrt{9 + x^2}}$.

SOLUTION Let $x = 3 \tan t, -\pi/2 < t < \pi/2$. Then $dx = 3 \sec^2 t\, dt$ and $\sqrt{9 + x^2} = 3 \sec t$.

$$\int \frac{dx}{\sqrt{9 + x^2}} = \int \frac{3 \sec^2 t}{3 \sec t}\, dt = \int \sec t\, dt$$

$$= \ln|\sec t + \tan t| + C$$

The last step, the integration of sec t, was handled in Problem 56 of Section 7.1. Now $\tan t = x/3$, which suggests the triangle in Figure 3, from which we conclude that $\sec t = \sqrt{9 + x^2}/3$. Thus,

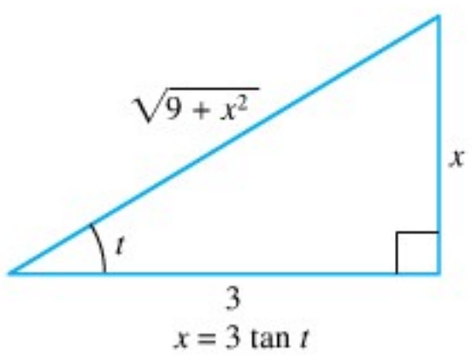


Figure 3

$$\int \frac{dx}{\sqrt{9 + x^2}} = \ln\left|\frac{\sqrt{9 + x^2} + x}{3}\right| + C$$

$$= \ln|\sqrt{9 + x^2} + x| - \ln 3 + C$$

$$= \ln|\sqrt{9 + x^2} + x| + K$$ ■

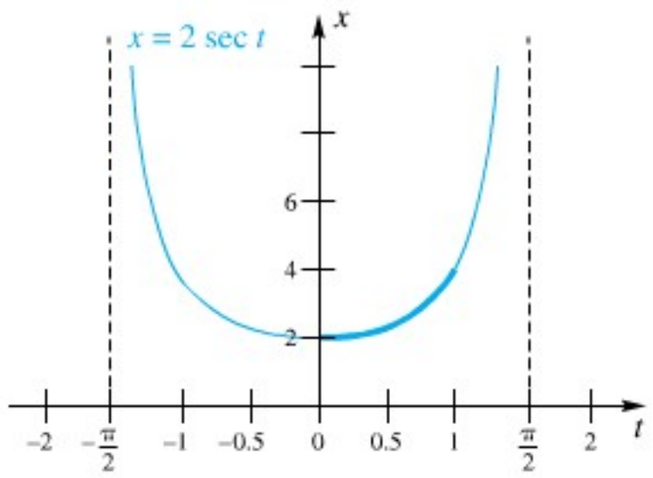


Figure 4

EXAMPLE 6 Calculate $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$.

SOLUTION Let $x = 2\sec t$, where $0 \le t < \pi/2$. Note that the restriction of t to this interval is acceptable, since x is in the interval $2 \le x \le 4$ (see Figure 4). This is important because it allows us to remove the absolute value sign that normally appears when we simplify $\sqrt{x^2-a^2}$. In our case,

$$\sqrt{x^2-4} = \sqrt{4\sec^2 t - 4} = \sqrt{4\tan^2 t} = 2|\tan t| = 2\tan t$$

We now use the theorem on substitution in a definite integral (which requires changing the limits of integration) to write

$$\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx = \int_0^{\pi/3} \frac{2\tan t}{2\sec t}\,2\sec t\tan t\,dt$$

$$= \int_0^{\pi/3} 2\tan^2 t\,dt = 2\int_0^{\pi/3} (\sec^2 t - 1)\,dt$$

$$= 2\big[\tan t - t\big]_0^{\pi/3} = 2\sqrt{3} - \frac{2\pi}{3} \approx 1.37$$ ■

Completing the Square When a quadratic expression of the type $x^2 + Bx + C$ appears under a radical, completing the square will prepare it for a trigonometric substitution.

EXAMPLE 7 Find (a) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+26}}$ and (b) $\displaystyle\int \frac{2x}{\sqrt{x^2+2x+26}}\,dx$.

SOLUTION

(a) $x^2 + 2x + 26 = x^2 + 2x + 1 + 25 = (x+1)^2 + 25$. Let $u = x + 1$ and $du = dx$. Then

$$\int \frac{dx}{\sqrt{x^2+2x+26}} = \int \frac{du}{\sqrt{u^2+25}}$$

Next let $u = 5\tan t$, $-\pi/2 < t < \pi/2$. Then $du = 5\sec^2 t\,dt$ and $\sqrt{u^2+25} = \sqrt{25(\tan^2 t + 1)} = 5\sec t$, so

$$\int \frac{du}{\sqrt{u^2+25}} = \int \frac{5\sec^2 t\,dt}{5\sec t} = \int \sec t\,dt$$

$$= \ln|\sec t + \tan t| + C$$

$$= \ln\left|\frac{\sqrt{u^2+25}}{5} + \frac{u}{5}\right| + C \qquad \text{(by Figure 5)}$$

$$= \ln|\sqrt{u^2+25} + u| - \ln 5 + C$$

$$= \ln|\sqrt{x^2+2x+26} + x + 1| + K$$

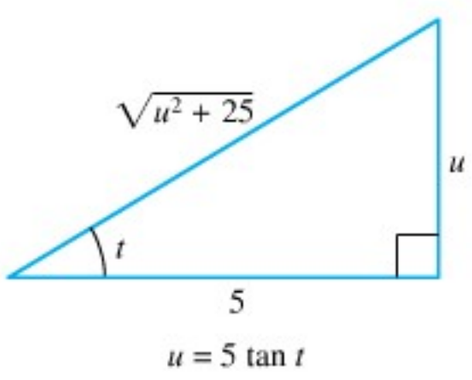


Figure 5

(b) To handle the second integral, we write

$$\int \frac{2x}{\sqrt{x^2+2x+26}}\,dx = \int \frac{2x+2}{\sqrt{x^2+2x+26}}\,dx - 2\int \frac{1}{\sqrt{x^2+2x+26}}\,dx$$

The first of the integrals on the right is handled by the substitution $u = x^2 + 2x + 26$; the second was just done. We obtain

$$\int \frac{2x}{\sqrt{x^2+2x+26}}\,dx =$$

$$2\sqrt{x^2+2x+26} - 2\ln|\sqrt{x^2+2x+26} + x + 1| + K$$ ■

Concepts Review

1. To handle $\int x\sqrt{x-3}\,dx$, make the substitution $u =$ _____.

2. To handle an integral involving $\sqrt{4-x^2}$, make the substitution $x =$ _____.

3. To handle an integral involving $\sqrt{4+x^2}$, make the substitution $x =$ _____.

4. To handle an integral involving $\sqrt{x^2-4}$, make the substitution $x =$ _____.

Problem Set 7.4

In Problems 1–16, perform the indicated integrations.

1. $\displaystyle\int x\sqrt{x+1}\,dx$

2. $\displaystyle\int x\sqrt[3]{x+\pi}\,dx$

3. $\displaystyle\int \frac{t\,dt}{\sqrt{3t+4}}$

4. $\displaystyle\int \frac{x^2+3x}{\sqrt{x+4}}\,dx$

5. $\displaystyle\int_1^2 \frac{dt}{\sqrt{t}+e}$

6. $\displaystyle\int_0^1 \frac{\sqrt{t}}{t+1}\,dt$

7. $\displaystyle\int t(3t+2)^{3/2}\,dt$

8. $\displaystyle\int x(1-x)^{2/3}\,dx$

9. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

10. $\displaystyle\int \frac{x^2\,dx}{\sqrt{16-x^2}}$

11. $\displaystyle\int \frac{dx}{(x^2+4)^{3/2}}$

12. $\displaystyle\int_2^3 \frac{dt}{t^2\sqrt{t^2-1}}$

13. $\displaystyle\int_{-2}^{-3} \frac{\sqrt{t^2-1}}{t^3}\,dt$

14. $\displaystyle\int \frac{t}{\sqrt{1-t^2}}\,dt$

15. $\displaystyle\int \frac{2z-3}{\sqrt{1-z^2}}\,dz$

16. $\displaystyle\int_0^{\pi} \frac{\pi x-1}{\sqrt{x^2+\pi^2}}\,dx$

In Problems 17–26, use the method of completing the square, along with a trigonometric substitution if needed, to evaluate each integral.

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+5}}$

19. $\displaystyle\int \frac{3x}{\sqrt{x^2+2x+5}}\,dx$

20. $\displaystyle\int \frac{2x-1}{\sqrt{x^2+4x+5}}\,dx$

21. $\displaystyle\int \sqrt{5-4x-x^2}\,dx$

22. $\displaystyle\int \frac{dx}{\sqrt{16+6x-x^2}}$

23. $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$

24. $\displaystyle\int \frac{x}{\sqrt{4x-x^2}}\,dx$

25. $\displaystyle\int \frac{2x+1}{x^2+2x+2}\,dx$

26. $\displaystyle\int \frac{2x-1}{x^2-6x+18}\,dx$

27. The region bounded by $y = 1/(x^2+2x+5)$, $y = 0$, $x = 0$, and $x = 1$, is revolved about the x-axis. Find the volume of the resulting solid.

28. The region of Problem 27 is revolved about the y-axis. Find the volume of the resulting solid.

29. Find $\displaystyle\int \frac{x\,dx}{x^2+9}$ by

(a) an algebraic substitution and

(b) a trigonometric substitution. Then reconcile your answers.

30. Find $\displaystyle\int_0^3 \frac{x^3\,dx}{\sqrt{9+x^2}}$ by making the substitutions

$$u = \sqrt{9+x^2},\quad u^2 = 9+x^2,\quad 2u\,du = 2x\,dx$$

31. Find $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$ by

(a) the substitution $u = \sqrt{4-x^2}$ and

(b) a trigonometric substitution. Then reconcile your answers.

Hint: $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$.

32. Two circles of radius b intersect as shown in Figure 6 with their centers $2a$ apart ($0 \le a \le b$). Find the area of the region of their overlap.

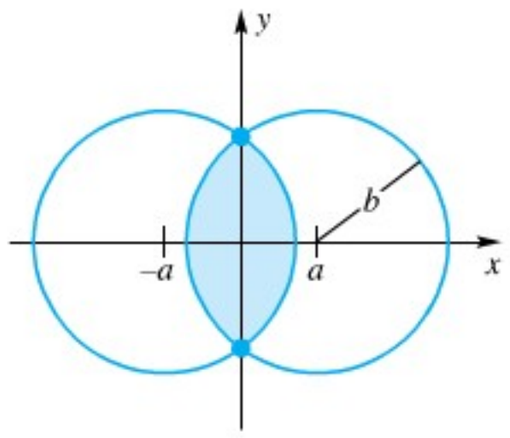


Figure 6

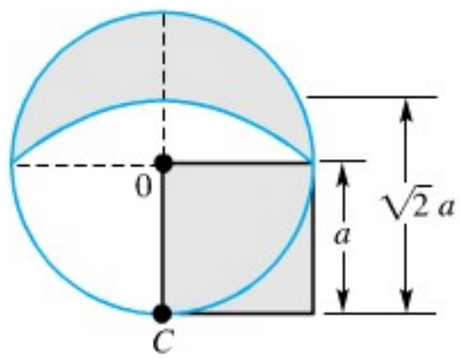


Figure 7

33. Hippocrates of Chios (ca. 430 B.C.) showed that the two shaded regions in Figure 7 have the same area (he squared the lune). Note that C is the center of the lower arc of the lune. Show Hippocrates' result

(a) using calculus and (b) without calculus.

34. Generalize the idea in Problem 33 by finding a formula for the area of the shaded lune shown in Figure 8.

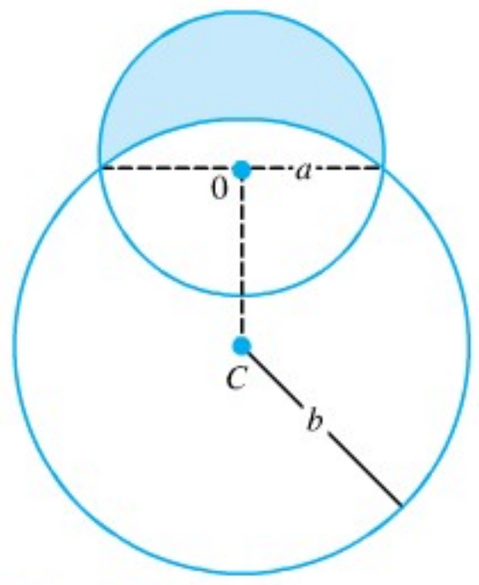


Figure 8

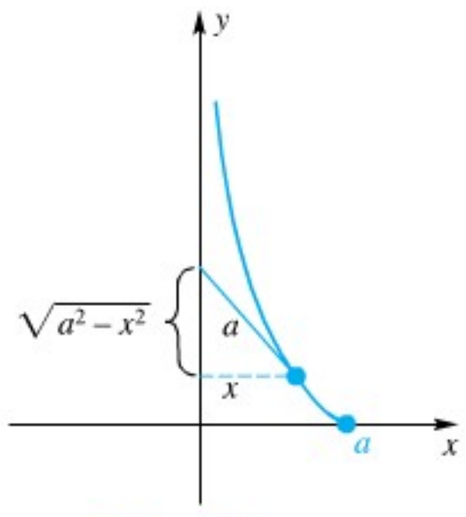


Figure 9

35. Starting at $(a, 0)$, an object is pulled along by a string of length a with the pulling end moving along the positive y-axis (Figure 9). The path of the object is a curve called a **tractrix**

and has the property that the string is always tangent to the curve. Set up a differential equation for the curve and solve it.

Answers to Concepts Review: **1.** $\sqrt{x-3}$ **2.** $2\sin t$ **3.** $2\tan t$ **4.** $2\sec t$

7.5 Integration of Rational Functions Using Partial Fractions

A **rational function** is by definition the quotient of two polynomial functions. Examples are

$$f(x) = \frac{2}{(x+1)^3}, \qquad g(x) = \frac{2x+2}{x^2-4x+8}, \qquad h(x) = \frac{x^5+2x^3-x+1}{x^3+5x}$$

Of these, f and g are **proper rational functions,** meaning that the degree of the numerator is less than that of the denominator. An improper (not proper) rational function can always be written as a sum of a polynomial function and a proper rational function. Thus, for example,

$$h(x) = \frac{x^5+2x^3-x+1}{x^3+5x} = x^2 - 3 + \frac{14x+1}{x^3+5x}$$

a result obtained by long division (Figure 1). Since polynomials are easy to integrate, the problem of integrating rational functions is really that of integrating proper rational functions. But can we always integrate proper rational functions? In theory, the answer is yes, though the practical details may be messy. Consider first the integrals of f and g above.

$$\begin{array}{r} x^2 - 3 \\ x^3+5x \;\overline{)\; x^5+2x^3 \;-\; x+1} \\ \underline{x^5+5x^3} \\ -3x^3 - x \\ \underline{-3x^3-15x} \\ 14x+1 \end{array}$$

Figure 1

EXAMPLE 1 Find $\displaystyle\int \frac{2}{(x+1)^3}\,dx$.

SOLUTION Think of the substitution $u = x + 1$.

$$\int \frac{2}{(x+1)^3}\,dx = 2\int (x+1)^{-3}\,dx = \frac{2(x+1)^{-2}}{-2} + C$$

$$= -\frac{1}{(x+1)^2} + C \qquad \blacksquare$$

EXAMPLE 2 Find $\displaystyle\int \frac{2x+2}{x^2-4x+8}\,dx$.

SOLUTION Think first of the substitution $u = x^2 - 4x + 8$ for which $du = (2x-4)\,dx$. Then write the given integral as a sum of two integrals.

$$\int \frac{2x+2}{x^2-4x+8}\,dx = \int \frac{2x-4}{x^2-4x+8}\,dx + \int \frac{6}{x^2-4x+8}\,dx$$

$$= \ln|x^2-4x+8| + 6\int \frac{1}{x^2-4x+8}\,dx$$

In the second integral, complete the square.

$$\int \frac{1}{x^2-4x+8}\,dx = \int \frac{1}{x^2-4x+4+4}\,dx = \int \frac{1}{(x-2)^2+4}\,dx$$

$$= \int \frac{1}{(x-2)^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$$

We conclude that

$$\int \frac{2x+2}{x^2-4x+8}\,dx = \ln|x^2-4x+8| + 3\tan^{-1}\left(\frac{x-2}{2}\right) + K \qquad \blacksquare$$

It is a remarkable fact that any proper rational function can be written as a sum of *simple* proper rational functions like those illustrated in Examples 1 and 2.

Partial Fraction Decomposition (Linear Factors) To add fractions is a standard algebraic exercise: find a common denominator and add. For example,

$$\frac{2}{x-1}+\frac{3}{x+1}=\frac{2(x+1)+3(x-1)}{(x-1)(x+1)}=\frac{5x-1}{(x-1)(x+1)}=\frac{5x-1}{x^2-1}$$

It is the reverse process of decomposing a fraction into a sum of simpler fractions that interests us now. We focus on the denominator and consider cases.

EXAMPLE 3 **Distinct Linear Factors** Decompose $(3x-1)/(x^2-x-6)$ and then find its indefinite integral.

SOLUTION Since the denominator factors as $(x+2)(x-3)$, it seems reasonable to hope for a decomposition of the following form:

$$\frac{3x-1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3} \tag{1}$$

Our job is, of course, to determine A and B so that (1) is an identity, a task that we find easier after we have multiplied both sides by $(x+2)(x-3)$. We obtain

$$3x-1=A(x-3)+B(x+2) \tag{2}$$

or, equivalently,

$$3x-1=(A+B)x+(-3A+2B) \tag{3}$$

However, (3) is an identity if and only if coefficients of like powers of x on both sides are equal; that is,

$$A+B=3$$
$$-3A+2B=-1$$

By solving this pair of equations for A and B, we obtain $A=\frac{7}{5}, B=\frac{8}{5}$. Consequently,

$$\frac{3x-1}{x^2-x-6}=\frac{3x-1}{(x+2)(x-3)}=\frac{\frac{7}{5}}{x+2}+\frac{\frac{8}{5}}{x-3}$$

and

$$\int\frac{3x-1}{x^2-x-6}\,dx=\frac{7}{5}\int\frac{1}{x+2}\,dx+\frac{8}{5}\int\frac{1}{x-3}\,dx$$
$$=\frac{7}{5}\ln|x+2|+\frac{8}{5}\ln|x-3|+C \qquad \blacksquare$$

If there was anything difficult about this process, it was the determination of A and B. We found their values by "brute force," but there is an easier way. In (2), which we wish to be an identity (that is, true for *every* value of x), substitute the convenient values $x=3$ and $x=-2$, obtaining

$$8=A\cdot 0+B\cdot 5$$
$$-7=A\cdot(-5)+B\cdot 0$$

This immediately gives $B=\frac{8}{5}$ and $A=\frac{7}{5}$.

You have just witnessed an odd, but correct, mathematical maneuver. Equation (1) turns out to be an identity (true for all x except -2 and 3) if and only if the essentially equivalent equation (2) is true precisely at -2 and 3. Ask yourself why

Solve This D.E.

"Often, there is little resemblance between a differential equation and its solution. Who would suppose that an expression as simple as

$$\frac{dy}{dx}=\frac{1}{a^2-x^2}$$

could be transformed into

$$y=\frac{1}{2a}\log_e\left(\frac{a+x}{a-x}\right)+C$$

This resembles the transformation of a chrysalis into a butterfly."

Silvanus P. Thompson

The method of partial fractions makes this an easy transformation. Do you see how it is done?

this is so. Ultimately it depends on the fact that the two sides of equation (2), both linear polynomials, are identical if they have the same values at any two points.

EXAMPLE 4 **Distinct Linear Factors** Find $\displaystyle\int \frac{5x+3}{x^3-2x^2-3x}\,dx$.

SOLUTION Since the denominator factors as $x(x+1)(x-3)$, we write

$$\frac{5x+3}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3}$$

and seek to determine A, B, and C. Clearing the fractions gives

$$5x+3 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1)$$

Substitution of the values $x = 0$, $x = -1$, and $x = 3$ results in

$$3 = A(-3)$$
$$-2 = B(4)$$
$$18 = C(12)$$

or $A = -1$, $B = -\frac{1}{2}$, $C = \frac{3}{2}$. Thus,

$$\int \frac{5x+3}{x^3-2x^2-3x}\,dx = -\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx$$
$$= -\ln|x| - \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-3| + C$$ ■

EXAMPLE 5 **Repeated Linear Factors** Find $\displaystyle\int \frac{x}{(x-3)^2}\,dx$.

SOLUTION Now the decomposition takes the form

$$\frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

with A and B to be determined. After clearing the fractions, we get

$$x = A(x-3) + B$$

If we now substitute the convenient value $x = 3$ and any other value, such as $x = 0$, we obtain $B = 3$ and $A = 1$. Thus,

$$\int \frac{x}{(x-3)^2}\,dx = \int \frac{1}{x-3}\,dx + 3\int \frac{1}{(x-3)^2}\,dx$$
$$= \ln|x-3| - \frac{3}{x-3} + C$$ ■

EXAMPLE 6 **Some Distinct, Some Repeated Linear Factors** Find

$$\int \frac{3x^2-8x+13}{(x+3)(x-1)^2}\,dx$$

SOLUTION We decompose the integrand in the following way:

$$\frac{3x^2-8x+13}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Clearing the fractions changes this to

$$3x^2-8x+13 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

Substitution of $x = 1$, $x = -3$, and $x = 0$ yields $C = 2$, $A = 4$, and $B = -1$. Thus,

$$\int \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2}\,dx = 4\int \frac{dx}{x+3} - \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2}$$

$$= 4\ln|x+3| - \ln|x-1| - \frac{2}{x-1} + C \quad \blacksquare$$

Be sure to note the inclusion of the two fractions $B/(x-1)$ and $C/(x-1)^2$ in the decomposition above. The general rule for decomposing fractions with repeated linear factors in the denominator is this: for each factor $(ax+b)^k$ of the denominator, there are k terms in the partial fraction decomposition:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}$$

Partial Fraction Decomposition (Quadratic Factors) In factoring the denominator of a fraction, we may well get some quadratic factors, such as $x^2 + 1$, that cannot be factored into linear factors without introducing complex numbers.

EXAMPLE 7 **A Single Quadratic Factor** Decompose $\dfrac{6x^2 - 3x + 1}{(4x+1)(x^2+1)}$ and then find its indefinite integral.

SOLUTION The best we can hope for is a decomposition of the form

$$\frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)} = \frac{A}{4x+1} + \frac{Bx + C}{x^2+1}$$

To determine the constants A, B, and C, we multiply both sides by $(4x+1)(x^2+1)$ and obtain

$$6x^2 - 3x + 1 = A(x^2+1) + (Bx + C)(4x+1)$$

Substitution of $x = -\frac{1}{4}$, $x = 0$, and $x = 1$ yields

$$\tfrac{6}{16} + \tfrac{3}{4} + 1 = A\left(\tfrac{17}{16}\right) \quad \Rightarrow \quad A = 2$$

$$1 = 2 + C \quad \Rightarrow \quad C = -1$$

$$4 = 4 + (B-1)5 \quad \Rightarrow \quad B = 1$$

Thus,

$$\int \frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)}\,dx = \int \frac{2}{4x+1}\,dx + \int \frac{x-1}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{4\,dx}{4x+1} + \frac{1}{2}\int \frac{2x\,dx}{x^2+1} - \int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln|4x+1| + \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + C \quad \blacksquare$$

EXAMPLE 8 **A Repeated Quadratic Factor** Find $\displaystyle\int \frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2}\,dx$.

SOLUTION Here the appropriate decomposition is

$$\frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2} = \frac{A}{x+3} + \frac{Bx + C}{x^2+2} + \frac{Dx + E}{(x^2+2)^2}$$

After considerable work, we discover that $A = 1, B = -1, C = 3, D = -5$, and $E = 0$. Thus,

$$\int \frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2}\,dx$$

$$= \int \frac{dx}{x+3} - \int \frac{x-3}{x^2+2}\,dx - 5\int \frac{x}{(x^2+2)^2}\,dx$$

$$= \int \frac{dx}{x+3} - \frac{1}{2}\int \frac{2x}{x^2+2}\,dx + 3\int \frac{dx}{x^2+2} - \frac{5}{2}\int \frac{2x\,dx}{(x^2+2)^2}$$

$$= \ln|x+3| - \frac{1}{2}\ln(x^2+2) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{5}{2(x^2+2)} + C \quad \blacksquare$$

Summary To decompose a rational function $f(x) = p(x)/q(x)$ into partial fractions, proceed as follows:

Step 1: If $f(x)$ is improper, that is, if $p(x)$ is of degree at least that of $q(x)$, divide $p(x)$ by $q(x)$, obtaining

$$f(x) = \text{a polynomial} + \frac{N(x)}{D(x)}$$

Step 2: Factor $D(x)$ into a product of linear and irreducible quadratic factors with real coefficients. By a theorem of algebra, this is always (theoretically) possible.

Step 3: For each factor of the form $(ax+b)^k$, expect the decomposition to have the terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

Step 4: For each factor of the form $(ax^2+bx+c)^m$, expect the decomposition to have the terms

$$\frac{B_1 x + C_1}{ax^2+bx+c} + \frac{B_2 x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_m x + C_m}{(ax^2+bx+c)^m}$$

Step 5: Set $N(x)/D(x)$ equal to the sum of all the terms found in Steps 3 and 4. The number of constants to be determined should equal the degree of the denominator, $D(x)$.

Step 6: Multiply both sides of the equation found in Step 5 by $D(x)$ and solve for the unknown constants. This can be done by either of two methods: (1) Equate coefficients of like-degree terms or (2) assign convenient values to the variable x.

The Logistic Differential Equation

In the last chapter, we saw that the assumption that the rate of growth of a population is proportional to its size, that is, $y' = ky$, leads to exponential growth. This assumption may be realistic until the available resources in the system are unable to sustain the population. In such a case, more reasonable assumptions are that there is a maximum capacity L that the system can sustain, and that the rate of growth is proportional to the product of the population size y and the "available room" $L - y$. These assumptions lead to the differential equation

$$y' = ky(L - y)$$

This is called the **logistic differential equation.** It is separable and now that we have covered the method of partial fractions, we can perform the necessary integration to solve it.

EXAMPLE 9 A population grows according to the logistic differential equation $y' = 0.0003y(2000 - y)$. The initial population size is 800. Solve this differential equation and use the solution to predict the population size at time $t = 2$.

≈ A Bound for the Answer

The initial population size is 800, and the rate of change in population size, y', is positive, so the population grows. As it nears 2000, the rate of change gets close to zero, so as $t \to \infty$, we have $y \to 2000$. The population at time $t = 2$ should be somewhere between 800 and 2000.

SOLUTION Writing y' as dy/dt, we see that the differential equation can be written as

$$\frac{dy}{dt} = 0.0003y(2000 - y)$$

$$\frac{dy}{y(2000 - y)} = 0.0003\,dt$$

$$\int \frac{dy}{y(2000 - y)} = \int 0.0003\,dt$$

The integral on the left can be evaluated using the method of partial fractions. We write

$$\frac{1}{y(2000 - y)} = \frac{A}{y} + \frac{B}{2000 - y}$$

which leads to

$$1 = A(2000 - y) + By$$

Substituting $y = 0$ and $y = 2000$ yields

$$1 = 2000A$$

$$1 = 2000B$$

Thus, $A = \dfrac{1}{2000}$ and $B = \dfrac{1}{2000}$, leading to

$$\int \left(\frac{1}{2000y} + \frac{1}{2000(2000 - y)} \right) dy = 0.0003t + C$$

$$\frac{1}{2000} \ln y - \frac{1}{2000} \ln(2000 - y) = 0.0003t + C$$

$$\ln \frac{y}{2000 - y} = 0.6t + 2000C$$

$$\frac{y}{2000 - y} = e^{0.6t + 2000C}$$

$$\frac{y}{2000 - y} = C_1 e^{0.6t}$$

Here, $C_1 = e^{2000C}$. At this point we can use the initial condition $y(0) = 800$ to determine C_1.

$$\frac{800}{2000 - 800} = C_1 e^{0.6 \cdot 0}$$

$$C_1 = \frac{800}{1200} = \frac{2}{3}$$

Thus

$$\frac{y}{2000 - y} = \frac{2}{3} e^{0.6t}$$

$$y = \frac{2}{3}(2000 - y)e^{0.6t}$$

$$y + \frac{2}{3} y e^{0.6t} = \frac{4000}{3} e^{0.6t}$$

$$y = \frac{(4000/3)e^{0.6t}}{1 + (2/3)e^{0.6t}} = \frac{4000/3}{2/3 + e^{-0.6t}}$$

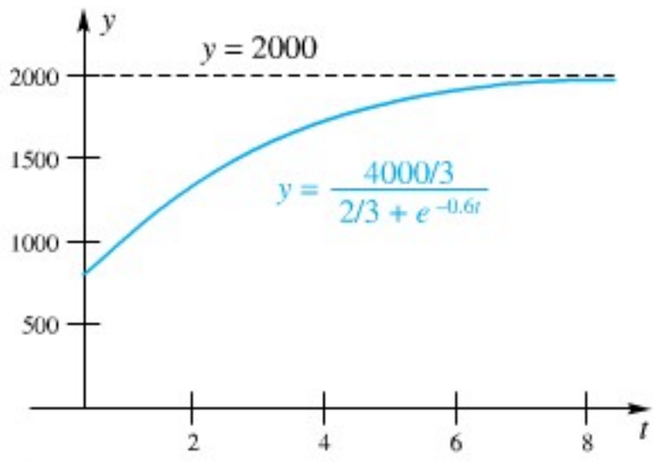


Figure 2

The population at time $t = 2$ is thus

$$y = \frac{4000/3}{2/3 + e^{-0.6 \cdot 2}} \approx 1378$$

A sketch of the population size as a function of t is given in Figure 2. ■

Concepts Review

1. If the degree of the polynomial $p(x)$ is less than the degree of $q(x)$, then $f(x) = p(x)/q(x)$ is called a _____ rational function.

2. To integrate the improper rational function $f(x) = (x^2 + 4)/(x + 1)$, we first rewrite it as $f(x) =$ _____.

3. If $(x - 1)(x + 1) + 3x + x^2 = ax^2 + bx + c$, then $a =$ _____, $b =$ _____, and $c =$ _____.

4. $(3x + 1)/[(x - 1)^2(x^2 + 1)]$ can be decomposed in the form _____.

Problem Set 7.5

In Problems 1–40, use the method of partial fraction decomposition to perform the required integration.

1. $\displaystyle\int \frac{1}{x(x+1)}\,dx$

2. $\displaystyle\int \frac{2}{x^2+3x}\,dx$

3. $\displaystyle\int \frac{3}{x^2-1}\,dx$

4. $\displaystyle\int \frac{5x}{2x^3+6x^2}\,dx$

5. $\displaystyle\int \frac{x-11}{x^2+3x-4}\,dx$

6. $\displaystyle\int \frac{x-7}{x^2-x-12}\,dx$

7. $\displaystyle\int \frac{3x-13}{x^2+3x-10}\,dx$

8. $\displaystyle\int \frac{x+\pi}{x^2-3\pi x+2\pi^2}\,dx$

9. $\displaystyle\int \frac{2x+21}{2x^2+9x-5}\,dx$

10. $\displaystyle\int \frac{2x^2-x-20}{x^2+x-6}\,dx$

11. $\displaystyle\int \frac{17x-3}{3x^2+x-2}\,dx$

12. $\displaystyle\int \frac{5-x}{x^2-x(\pi+4)+4\pi}\,dx$

13. $\displaystyle\int \frac{2x^2+x-4}{x^3-x^2-2x}\,dx$

14. $\displaystyle\int \frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}\,dx$

15. $\displaystyle\int \frac{6x^2+22x-23}{(2x-1)(x^2+x-6)}\,dx$

16. $\displaystyle\int \frac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32}\,dx$

17. $\displaystyle\int \frac{x^3}{x^2+x-2}\,dx$

18. $\displaystyle\int \frac{x^3+x^2}{x^2+5x+6}\,dx$

19. $\displaystyle\int \frac{x^4+8x^2+8}{x^3-4x}\,dx$

20. $\displaystyle\int \frac{x^6+4x^3+4}{x^3-4x^2}\,dx$

21. $\displaystyle\int \frac{x+1}{(x-3)^2}\,dx$

22. $\displaystyle\int \frac{5x+7}{x^2+4x+4}\,dx$

23. $\displaystyle\int \frac{3x+2}{x^3+3x^2+3x+1}\,dx$

24. $\displaystyle\int \frac{x^6}{(x-2)^2(1-x)^5}\,dx$

25. $\displaystyle\int \frac{3x^2-21x+32}{x^3-8x^2+16x}\,dx$

26. $\displaystyle\int \frac{x^2+19x+10}{2x^4+5x^3}\,dx$

27. $\displaystyle\int \frac{2x^2+x-8}{x^3+4x}\,dx$

28. $\displaystyle\int \frac{3x+2}{x(x+2)^2+16x}\,dx$

29. $\displaystyle\int \frac{2x^2-3x-36}{(2x-1)(x^2+9)}\,dx$

30. $\displaystyle\int \frac{1}{x^4-16}\,dx$

31. $\displaystyle\int \frac{1}{(x-1)^2(x+4)^2}\,dx$

32. $\displaystyle\int \frac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}\,dx$

33. $\displaystyle\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)}\,dt$

34. $\displaystyle\int \frac{\cos t}{\sin^4 t - 16}\,dt$

35. $\displaystyle\int \frac{x^3-4x}{(x^2+1)^2}\,dx$

36. $\displaystyle\int \frac{(\sin t)(4\cos^2 t - 1)}{(\cos t)(1 + 2\cos^2 t + \cos^4 t)}\,dt$

37. $\displaystyle\int \frac{2x^3+5x^2+16x}{x^5+8x^3+16x}\,dx$

38. $\displaystyle\int_4^6 \frac{x-17}{x^2+x-12}\,dx$

39. $\displaystyle\int_0^{\pi/4} \frac{\cos\theta}{(1-\sin^2\theta)(\sin^2\theta+1)^2}\,d\theta$

40. $\displaystyle\int_1^5 \frac{3x+13}{x^2+4x+3}\,dx$

In Problems 41–44, solve the logistic differential equation representing population growth with the given initial condition. Then use the solution to predict the population size at time $t = 3$.

41. $y' = y(1 - y), y(0) = 0.5$

42. $y' = \frac{1}{10}y(12 - y), y(0) = 2$

43. $y' = 0.0003y\,(8000 - y), y(0) = 1000$

44. $y' = 0.001y\,(4000 - y), y(0) = 100$

45. Solve the logistic differential equation for an arbitrary constant of proportionality k, capacity L, and initial condition $y(0) = y_0$.

46. Explain what happens to the solution of the logistic differential equation if the initial population size is *larger* than the maximum capacity.

47. Without solving the logistic equation or referring to its solution, explain how you know that if $y_0 < L$, then the population size is increasing.

48. Assuming $y_0 < L$, for what values of t is the graph of the population size $y(t)$ concave up?

49. Suppose that the earth will not support a population of more than 16 billion and that there were 2 billion people in 1925 and 4 billion people in 1975. Then, if y is the population t years after 1925, an appropriate model is the logistic differential equation

$$\frac{dy}{dt} = ky(16 - y)$$

(a) Solve this differential equation.

(b) Predict the population in 2015.

(c) When will the population be 9 billion?

50. Do Problem 49 assuming that the upper limit for the population is 10 billion.

51. The Law of Mass Action in chemistry results in the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x), \qquad k > 0, \quad a > 0, \quad b > 0$$

where x is the amount of a substance at time t resulting from the reaction of two others. Assume that $x = 0$ when $t = 0$.

(a) Solve this differential equation in the case $b > a$.

(b) Show that $x \to a$ as $t \to \infty$ (if $b > a$).

(c) Suppose that $a = 2$ and $b = 4$, and that 1 gram of the substance is formed in 20 minutes. How much will be present in 1 hour?

(d) Solve the differential equation if $a = b$.

52. The differential equation

$$\frac{dy}{dt} = k(y - m)(M - y)$$

with $k > 0$ and $0 \le m < y_0 < M$ is used to model some growth problems. Solve the equation and find $\lim_{t \to \infty} y$.

53. As a model for the production of trypsin from trypsinogen in digestion, biochemists have proposed the model

$$\frac{dy}{dt} = k(A - y)(B + y)$$

where $k > 0$, A is the initial amount of trypsinogen, and B is the original amount of trypsin. Solve this differential equation.

54. Evaluate

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(\sin^2 x + 1)^2}\,dx$$

Answers to Concepts Review: **1.** proper **2.** $x - 1 + \frac{5}{x + 1}$ **3.** $2; 3; -1$ **4.** $\frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 1}$

CHAPTER 5 Applications of the Integral

5.1 The Area of a Plane Region
5.2 Volumes of Solids: Slabs, Disks, Washers
5.3 Volumes of Solids of Revolution: Shells
5.4 Length of a Plane Curve
5.5 Work and Fluid Force
5.6 Moments and Center of Mass
5.7 Probability and Random Variables

5.1 The Area of a Plane Region

The brief discussion of area in Section 4.1 served to motivate the definition of the definite integral. With the latter notion now firmly established, we use the definite integral to calculate areas of regions of more and more complicated shapes. As is our practice, we begin with simple cases.

A Region above the x-Axis Let $y = f(x)$ determine a curve in the xy-plane and suppose that f is continuous and nonnegative on the interval $a \le x \le b$ (as in Figure 1). Consider the region R bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and $y = 0$. We refer to R as the region under $y = f(x)$ between $x = a$ and $x = b$. Its area $A(R)$ is given by

$$A(R) = \int_a^b f(x)\,dx$$

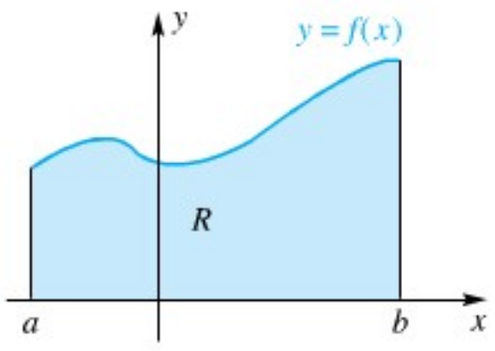


Figure 1

EXAMPLE 1 Find the area of the region R under $y = x^4 - 2x^3 + 2$ between $x = -1$ and $x = 2$.

SOLUTION The graph of R is shown in Figure 2. A reasonable estimate for the area of R is its base times an average height, say $(3)(2) = 6$. The exact value is

$$A(R) = \int_{-1}^{2} (x^4 - 2x^3 + 2)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{2} + 2x\right]_{-1}^{2}$$
$$= \left(\frac{32}{5} - \frac{16}{2} + 4\right) - \left(-\frac{1}{5} - \frac{1}{2} - 2\right) = \frac{51}{10} = 5.1$$

The calculated value 5.1 is close enough to our estimate, 6, to give us confidence in its correctness. ■

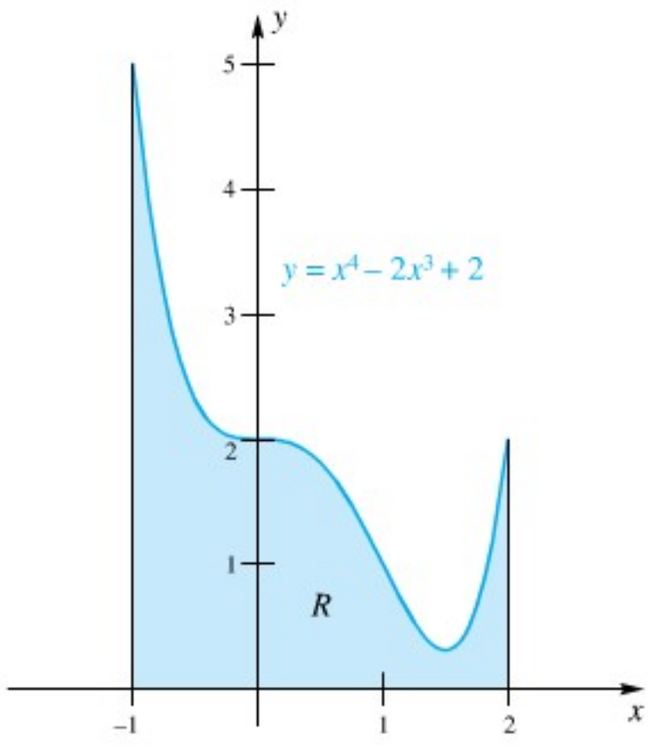


Figure 2

A Region Below the x-Axis Area is a nonnegative number. If the graph of $y = f(x)$ is below the x-axis, then $\int_a^b f(x)\,dx$ is a negative number and therefore cannot be an area. However, it is just the negative of the area of the region bounded by $y = f(x)$, $x = a$, $x = b$, and $y = 0$.

EXAMPLE 2 Find the area of the region R bounded by $y = x^2/3 - 4$, the x-axis, $x = -2$, and $x = 3$.

SOLUTION The region R is shown in Figure 3. Our preliminary estimate for its area is $(5)(3) = 15$. The exact value is

$$A(R) = -\int_{-2}^{3}\left(\frac{x^2}{3} - 4\right)dx = \int_{-2}^{3}\left(-\frac{x^2}{3} + 4\right)dx$$
$$= \left[-\frac{x^3}{9} + 4x\right]_{-2}^{3} = \left(-\frac{27}{9} + 12\right) - \left(\frac{8}{9} - 8\right) = \frac{145}{9} \approx 16.11$$

We are reassured by the nearness of 16.11 to our estimate. ■

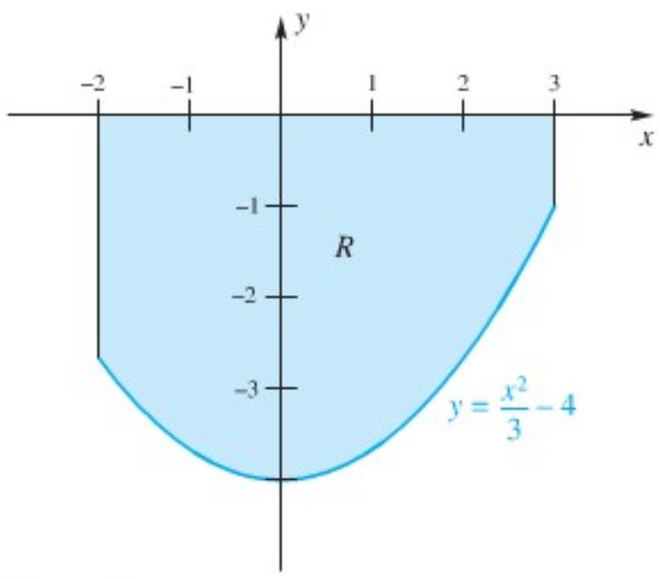


Figure 3

EXAMPLE 3 Find the area of the region R bounded by $y = x^3 - 3x^2 - x + 3$, the segment of the x-axis between $x = -1$ and $x = 2$, and the line $x = 2$.

SOLUTION The region R is shaded in Figure 4. Note that part of it is above the x-axis and part is below. The areas of these two parts, R_1 and R_2, must be calculated separately. You can check that the curve crosses the x-axis at -1, 1, and 3. Thus,

$$\begin{aligned} A(R) &= A(R_1) + A(R_2) \\ &= \int_{-1}^{1} (x^3 - 3x^2 - x + 3)\,dx - \int_{1}^{2} (x^3 - 3x^2 - x + 3)\,dx \\ &= \left[\frac{x^4}{4} - x^3 - \frac{x^2}{2} + 3x\right]_{-1}^{1} - \left[\frac{x^4}{4} - x^3 - \frac{x^2}{2} + 3x\right]_{1}^{2} \\ &= 4 - \left(-\frac{7}{4}\right) = \frac{23}{4} \end{aligned}$$

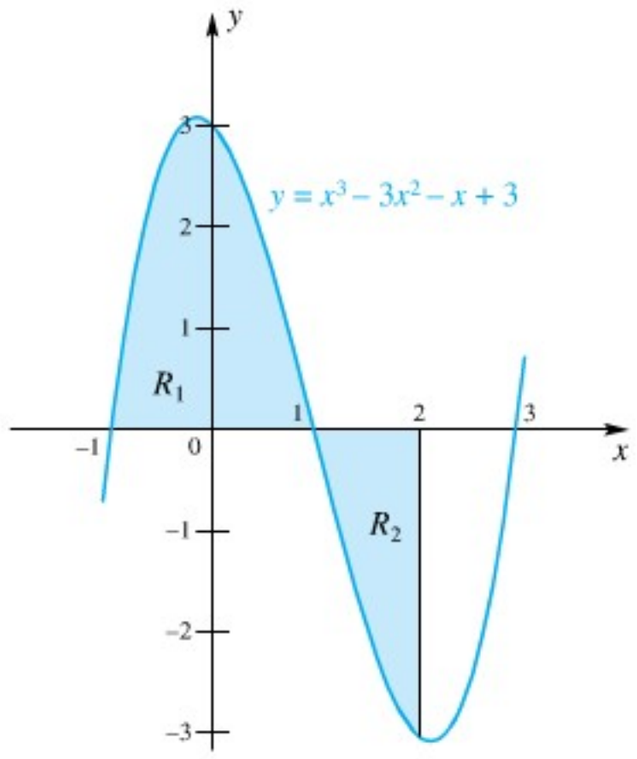


Figure 4

Notice that we could have written this area as one integral using the absolute value symbol,

$$A(R) = \int_{-1}^{2} |x^3 - 3x^2 - x + 3|\,dx$$

but this is no real simplification since, in order to evaluate this integral, we would have to split it into two parts, just as we did above. ■

A Helpful Way of Thinking For simple regions of the type considered above, it is quite easy to write down the correct integral. When we consider more complicated regions (e.g., regions between two curves), the task of selecting the right integral is more difficult. However, there is a way of thinking that can be very helpful. It goes back to the definition of area and of the definite integral. Here it is in five steps.

Step 1: Sketch the region.

Step 2: Slice it into thin pieces (strips); label a typical piece.

Step 3: Approximate the area of this typical piece as if it were a rectangle.

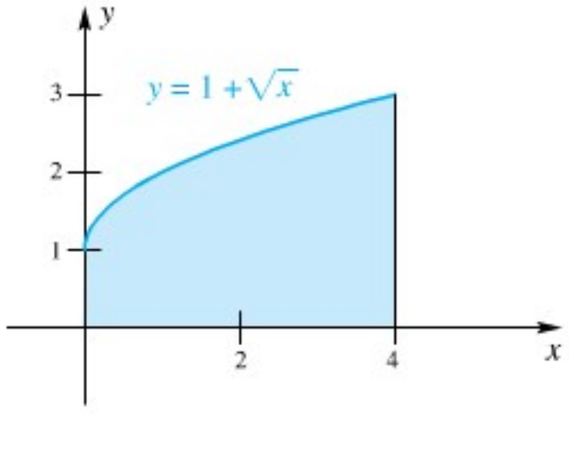
Step 4: Add up the approximations to the areas of the pieces.

Step 5: Take the limit as the width of the pieces approaches zero, thus getting a definite integral.

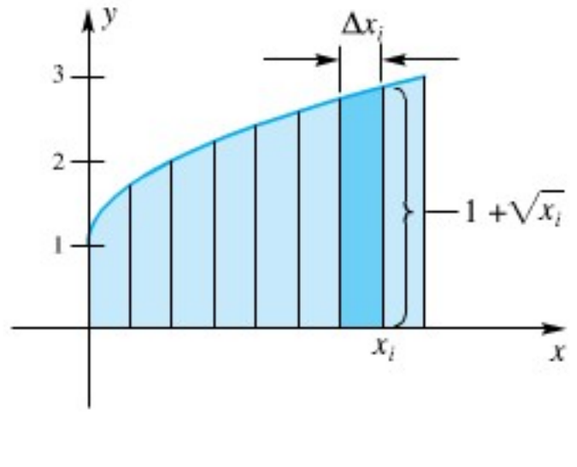
To illustrate, we consider yet another simple example.

EXAMPLE 4 Set up the integral for the area of the region under $y = 1 + \sqrt{x}$ between $x = 0$ and $x = 4$ (Figure 5).

1. Sketch



2. Slice



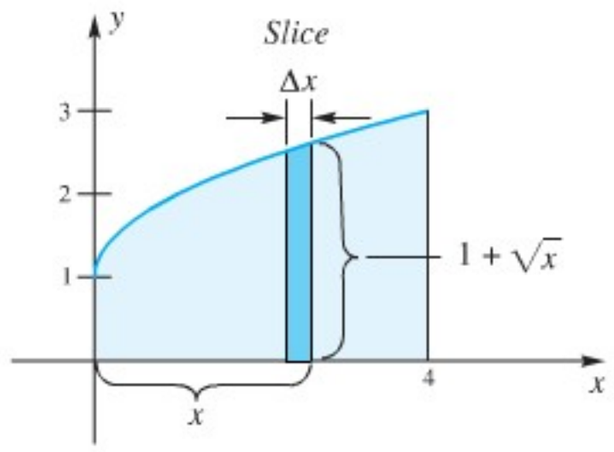
3. Approximate area of typical piece:
$\Delta A_i \approx (1 + \sqrt{x_i})\,\Delta x_i$

4. Add up: $A \approx \sum_{i=1}^{n} (1 + \sqrt{x_i})\,\Delta x_i$

5. Take limit: $A = \int_0^4 (1 + \sqrt{x})\,dx$

Figure 5

SOLUTION Once we understand this five-step procedure, we can abbreviate it to three: *slice*, *approximate*, *integrate*. Think of the word *integrate* as incorporating two steps: (1) add the areas of the pieces and (2) take the limit as the piece width tends to zero. In this process, $\Sigma \ldots \Delta x$ transforms into $\int \ldots dx$ as we take the limit. Figure 6 gives the abbreviated form for the same problem.



Approximate

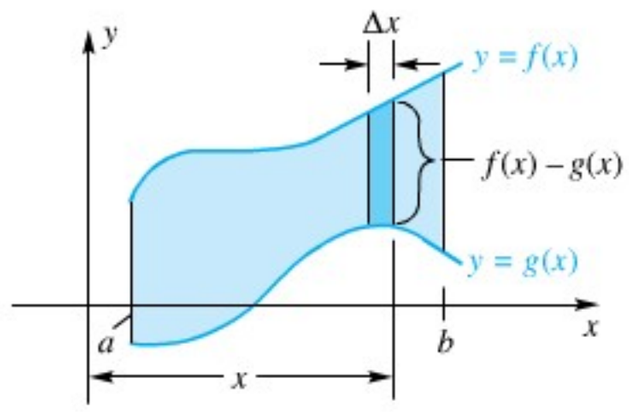
$$\Delta A \approx (1 + \sqrt{x})\,\Delta x$$

Integrate

$$A = \int_0^4 (1 + \sqrt{x})\,dx$$

Figure 6 ■

A Region Between Two Curves Consider curves $y = f(x)$ and $y = g(x)$ with $g(x) \le f(x)$ on $a \le x \le b$. They determine the region shown in Figure 7. We use the *slice*, *approximate*, *integrate* method to find its area. Be sure to note that $f(x) - g(x)$ gives the correct height for the thin slice, even when the graph of g goes below the x-axis. In this case $g(x)$ is negative; so subtracting $g(x)$ is the same as adding a positive number. You can check that $f(x) - g(x)$ also gives the correct height, even when both $f(x)$ and $g(x)$ are negative.



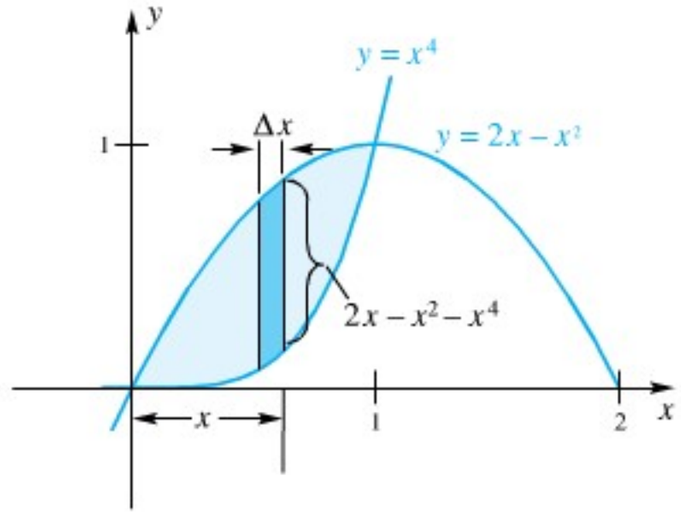
$$\Delta A \approx [f(x) - g(x)]\,\Delta x$$

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Figure 7

EXAMPLE 5 Find the area of the region between the curves $y = x^4$ and $y = 2x - x^2$.

SOLUTION We start by finding where the two curves intersect. To do this, we need to solve $2x - x^2 = x^4$, a fourth-degree equation, which would usually be difficult to solve. However, in this case $x = 0$ and $x = 1$ are rather obvious solutions. Our sketch of the region, together with the appropriate approximation and the corresponding integral, is shown in Figure 8.



$$\Delta A \approx (2x - x^2 - x^4)\,\Delta x$$

$$A = \int_0^1 (2x - x^2 - x^4)\,dx$$

Figure 8

One job remains: to evaluate the integral.

$$\int_0^1 (2x - x^2 - x^4)\,dx = \left[x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 1 - \frac{1}{3} - \frac{1}{5} = \frac{7}{15}$$ ■

EXAMPLE 6 **Horizontal Slicing** Find the area of the region between the parabola $y^2 = 4x$ and the line $4x - 3y = 4$.

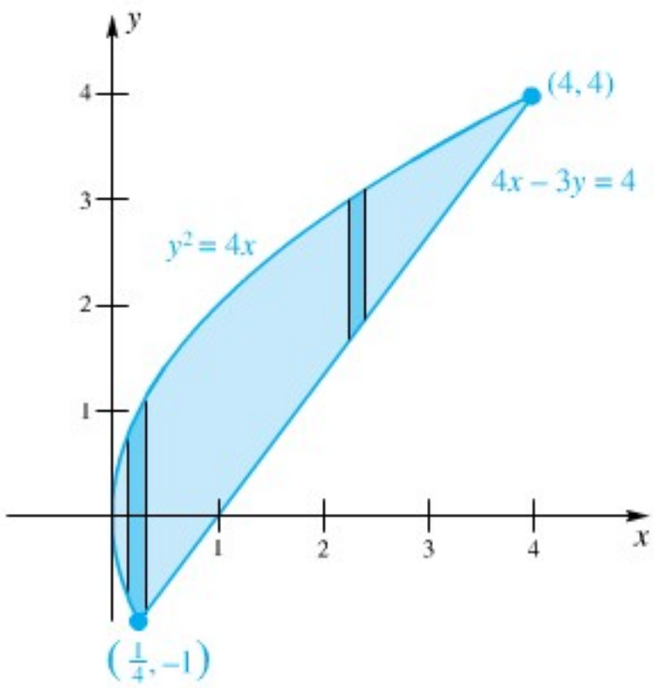


Figure 9

SOLUTION We will need the points of intersection of these two curves. The y-coordinates of these points can be found by writing the second equation as $4x = 3y + 4$ and then equating the two expressions for $4x$.

$$\begin{aligned} y^2 &= 3y + 4 \\ y^2 - 3y - 4 &= 0 \\ (y - 4)(y + 1) &= 0 \\ y &= 4, -1 \end{aligned}$$

When $y = 4$, $x = 4$ and when $y = -1$, $x = \frac{1}{4}$, so we conclude that the points of intersection are (4, 4) and $\left(\frac{1}{4}, -1\right)$. The region between the curves is shown in Figure 9.

Now imagine slicing this region vertically. We face a problem, because the lower boundary consists of two different curves. Slices at the extreme left extend from the lower branch of the parabola to its upper branch. For the rest of the region, slices extend from the line to the parabola. To solve the problem with vertical slices requires that we first split our region into two parts, set up an integral for each part, and then evaluate both integrals.

A far better approach is to slice the region horizontally as shown in Figure 10, thus using y rather than x as the integration variable. Note that horizontal slices always go from the parabola (at the left) to the line (at the right). The length of such a slice is the larger x-value $\left(x = \frac{1}{4}(3y + 4)\right)$ minus the smaller x-value $\left(x = \frac{1}{4}y^2\right)$.

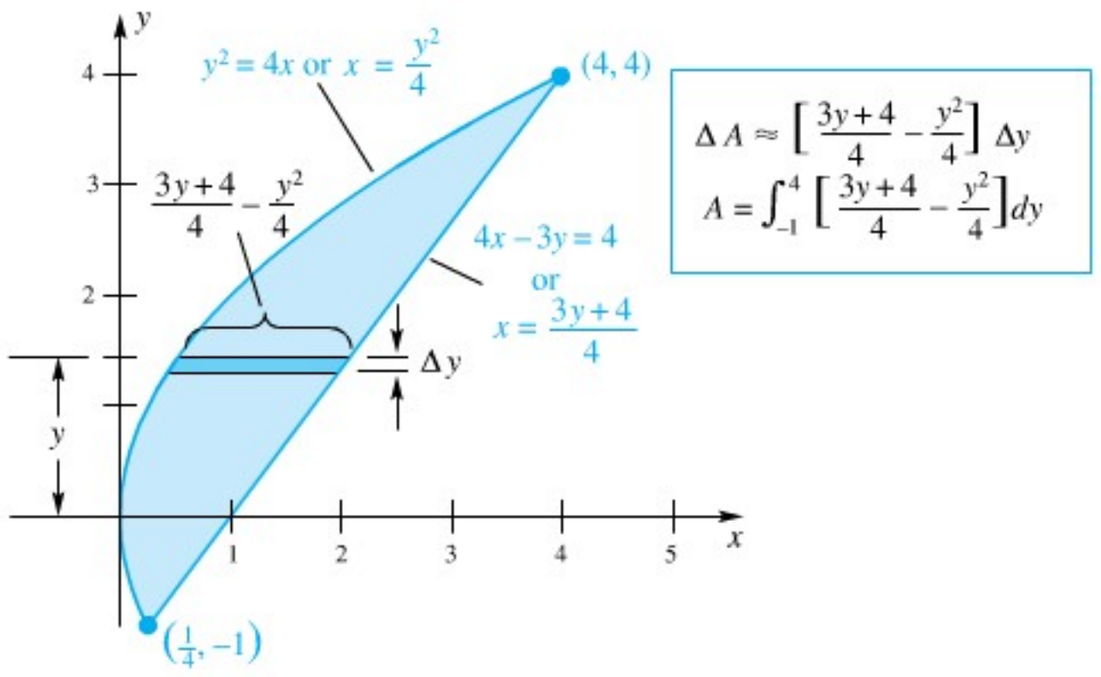


Figure 10

$$A = \int_{-1}^{4}\left[\frac{3y + 4 - y^2}{4}\right] dy = \frac{1}{4}\int_{-1}^{4}(3y + 4 - y^2)\,dy$$

$$= \frac{1}{4}\left[\frac{3y^2}{2} + 4y - \frac{y^3}{3}\right]_{-1}^{4}$$

$$= \frac{1}{4}\left[\left(24 + 16 - \frac{64}{3}\right) - \left(\frac{3}{2} - 4 + \frac{1}{3}\right)\right]$$

$$= \frac{125}{24} \approx 5.21$$

There are two items to note: (1) The integrand resulting from a horizontal slicing involves y, not x; and (2) to get the integrand, solve both equations for x and subtract the smaller x-value from the larger. ■

Distance and Displacement Consider an object moving along a straight line with velocity $v(t)$ at time t. If $v(t) \geq 0$, then $\int_a^b v(t)\,dt$ gives the distance traveled during the time interval $a \leq t \leq b$. However, if $v(t)$ is sometimes negative (which corresponds to the object moving in reverse), then

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

measures the **displacement** of the object, that is, the directed distance from its starting position $s(a)$ to its ending position $s(b)$. To get the **total distance** that the object traveled during $a \leq t \leq b$, we must calculate $\int_a^b |v(t)|\,dt$, the area between the velocity curve and the t-axis.

EXAMPLE 7 An object is at position $s = 3$ at time $t = 0$. Its velocity at time t is $v(t) = 5 \sin 6\pi t$. What is the position of the object at time $t = 2$, and how far did it travel during this time?

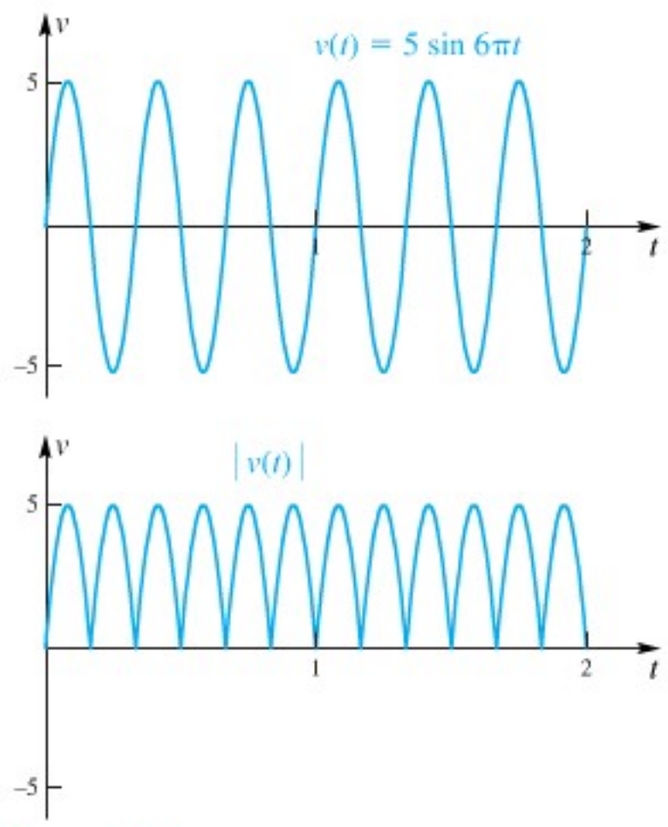


Figure 11

SOLUTION The object's displacement, that is, change in position, is

$$s(2) - s(0) = \int_0^2 v(t)\,dt = \int_0^2 5 \sin 6\pi t\,dt = \left[-\frac{5}{6\pi} \cos 6\pi t\right]_0^2 = 0$$

Thus, $s(2) = s(0) + 0 = 3 + 0 = 3$. The object is at position 3 at time $t = 2$. The total distance traveled is

$$\int_0^2 |v(t)|\,dt = \int_0^2 |5 \sin 6\pi t|\,dt$$

To perform this integration we make use of symmetry (see Figure 11). Thus

$$\int_0^2 |v(t)|\,dt = 12 \int_0^{2/12} 5 \sin 6\pi t\,dt = 60\left[-\frac{1}{6\pi} \cos 6\pi t\right]_0^{1/6} = \frac{20}{\pi} \approx 6.3662$$ ■

Concepts Review

1. Let R be the region between the curve $y = f(x)$ and the x-axis on the interval $[a, b]$. If $f(x) \geq 0$ for all x in $[a, b]$, then $A(R) =$ _____, but if $f(x) \leq 0$ for all x in $[a, b]$, then $A(R) =$ _____.

2. To find the area of the region between two curves, it is wise to think of the following three-word motto: _____.

3. Suppose that the curves $y = f(x)$ and $y = g(x)$ bound a region R on which $f(x) \leq g(x)$. Then the area of R is given by $A(R) = \int_a^b$ _____ dx, where a and b are determined by solving the equation _____.

4. If $p(y) \leq q(y)$ for all y in $[c, d]$, then the area $A(R)$ of the region R bounded by the curves $x = p(y)$ and $x = q(y)$ between $y = c$ and $y = d$ is given by $A(R) =$ _____.

Problem Set 5.1

In Problems 1–10, use the three-step procedure (slice, approximate, integrate) to set up and evaluate an integral (or integrals) for the area of the indicated region.

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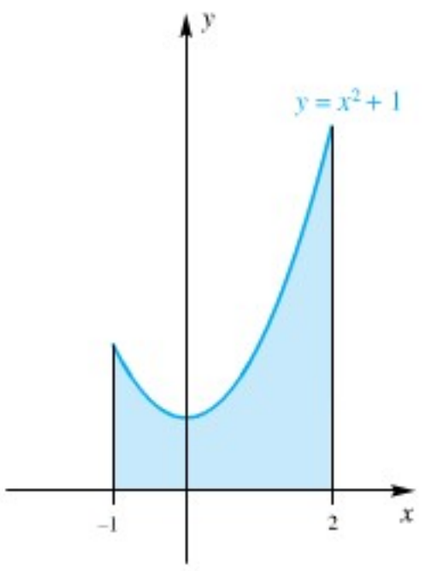

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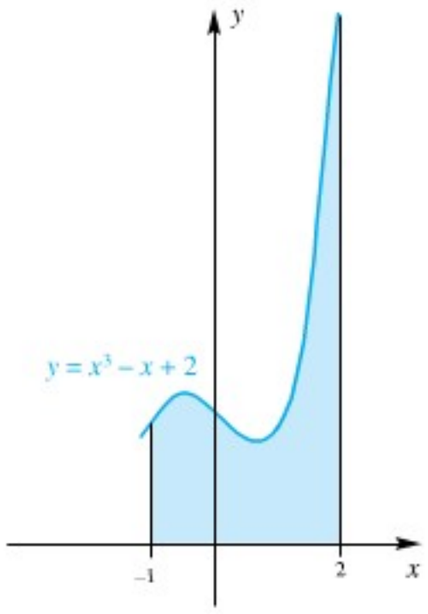

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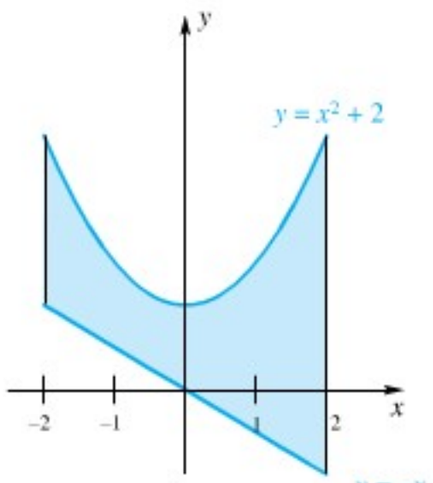

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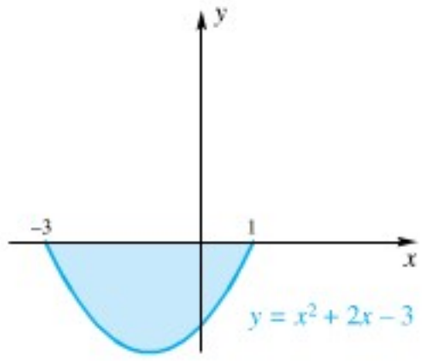

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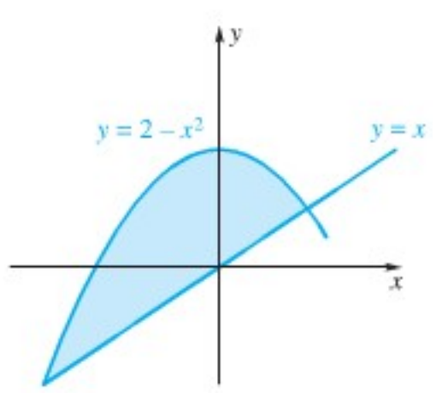

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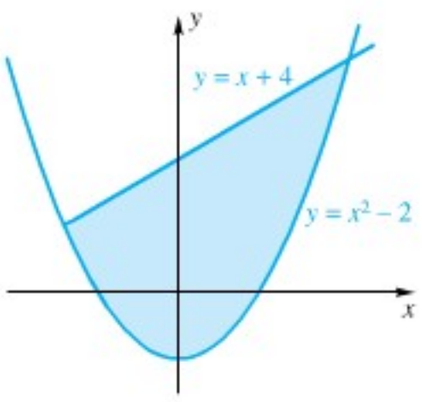

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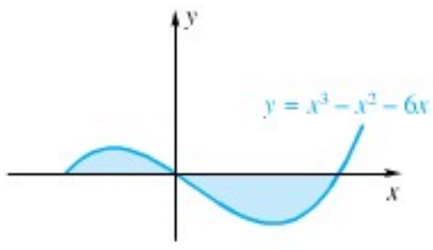

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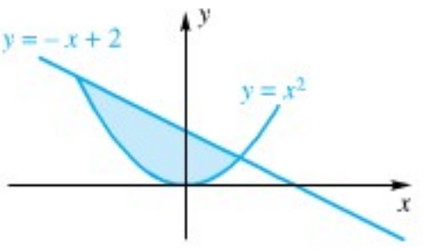

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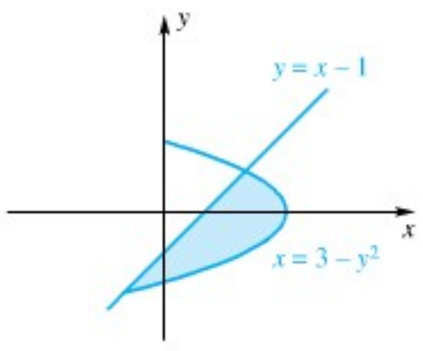

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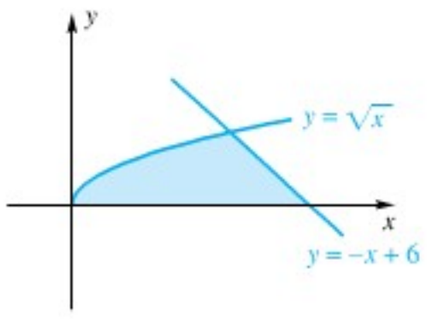


≈ *In Problems 11–28, sketch the region bounded by the graphs of the given equations, show a typical slice, approximate its area, set up an integral, and calculate the area of the region. Make an estimate of the area to confirm your answer.*

11. $y = 3 - \frac{1}{3}x^2$, $y = 0$, between $x = 0$ and $x = 3$

12. $y = 5x - x^2$, $y = 0$, between $x = 1$ and $x = 3$

13. $y = (x - 4)(x + 2)$, $y = 0$, between $x = 0$ and $x = 3$

14. $y = x^2 - 4x - 5$, $y = 0$, between $x = -1$ and $x = 4$

15. $y = \frac{1}{4}(x^2 - 7)$, $y = 0$, between $x = 0$ and $x = 2$

16. $y = x^3$, $y = 0$, between $x = -3$ and $x = 3$

17. $y = \sqrt[3]{x}$, $y = 0$, between $x = -2$ and $x = 2$

18. $y = \sqrt{x} - 10$, $y = 0$, between $x = 0$ and $x = 9$

19. $y = (x - 3)(x - 1)$, $y = x$

20. $y = \sqrt{x}$, $y = x - 4$, $x = 0$

21. $y = x^2 - 2x$, $y = -x^2$

22. $y = x^2 - 9$, $y = (2x - 1)(x + 3)$

23. $x = 8y - y^2$, $x = 0$

24. $x = (3 - y)(y + 1)$, $x = 0$

25. $x = -6y^2 + 4y$, $x + 3y - 2 = 0$

26. $x = y^2 - 2y$, $x - y - 4 = 0$

27. $4y^2 - 2x = 0$, $4y^2 + 4x - 12 = 0$

28. $x = 4y^4$, $x = 8 - 4y^4$

29. Sketch the region R bounded by $y = x + 6$, $y = x^3$, and $2y + x = 0$. Then find its area. *Hint:* Divide R into two pieces.

30. Find the area of the triangle with vertices at $(-1, 4)$, $(2, -2)$, and $(5, 1)$ by integration.

31. An object moves along a line so that its velocity at time t is $v(t) = 3t^2 - 24t + 36$ feet per second. Find the displacement and total distance traveled by the object for $-1 \le t \le 9$.

32. Follow the directions of Problem 31 if $v(t) = \frac{1}{2} + \sin 2t$ and the interval is $0 \le t \le 3\pi/2$.

33. Starting at $s = 0$ when $t = 0$, an object moves along a line so that its velocity at time t is $v(t) = 2t - 4$ centimeters per second. How long will it take to get to $s = 12$? To travel a total distance of 12 centimeters?

34. Consider the curve $y = 1/x^2$ for $1 \le x \le 6$.

(a) Calculate the area under this curve.

(b) Determine c so that the line $x = c$ bisects the area of part (a).

(c) Determine d so that the line $y = d$ bisects the area of part (a).

35. Calculate areas A, B, C, and D in Figure 12. Check by calculating $A + B + C + D$ in one integration.

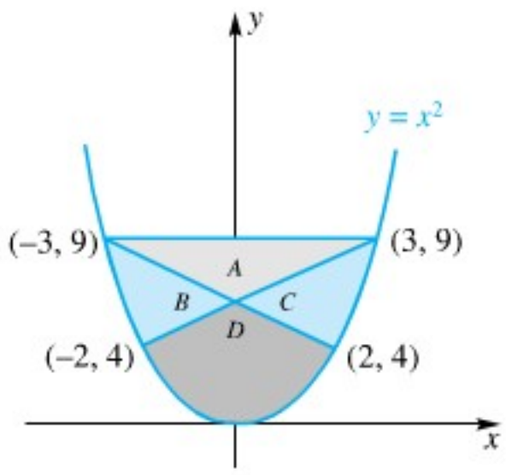


Figure 12

36. Prove Cavalieri's Principle. (Bonaventura Cavalieri (1598–1647) developed this principle in 1635.) If two regions have the same height at every x in $[a, b]$, then they have the same area (see Figure 13).

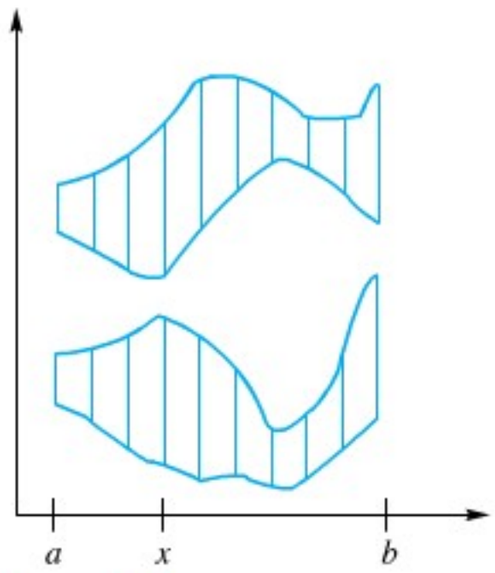


Figure 13

37. Use Cavalieri's Principle (not integration; see Problem 36) to show that the shaded regions in Figure 14 have the same area.

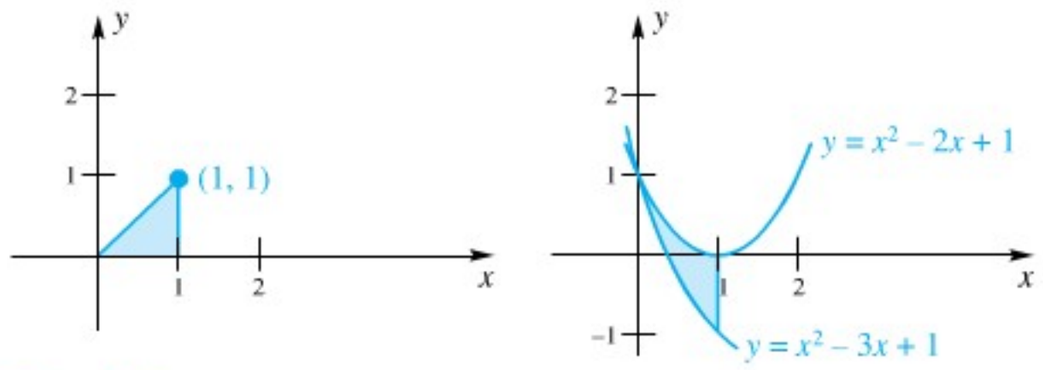


Figure 14

38. Find the area of the region trapped between $y = \sin x$ and $y = \frac{1}{2}$, $0 \le x \le 17\pi/6$.

Answers to Concepts Review: **1.** $\int_a^b f(x)\,dx$; $-\int_a^b f(x)\,dx$ **2.** slice, approximate, integrate **3.** $[g(x) - f(x)]$; $f(x) = g(x)$ **4.** $\int_c^d [q(y) - p(y)]\,dy$

5.2 Volumes of Solids: Slabs, Disks, Washers

That the definite integral can be used to calculate *areas* is not surprising; it was invented for that purpose. But uses of the integral go far beyond that application. Many quantities can be thought of as a result of slicing something into small pieces, approximating each piece, adding up, and taking the limit as the pieces shrink in size. This method of slice, approximate, and integrate can be used to find the *volumes* of solids provided that the volume of each slice is easy to approximate.

What is volume? We start with simple solids called *right cylinders*, four of which are shown in Figure 1. In each case, the solid is generated by moving a plane region (the base) through a distance h in a direction perpendicular to that region. And in each case, the volume of the solid is defined to be the area A of the base times the height h; that is,

$$V = A \cdot h$$

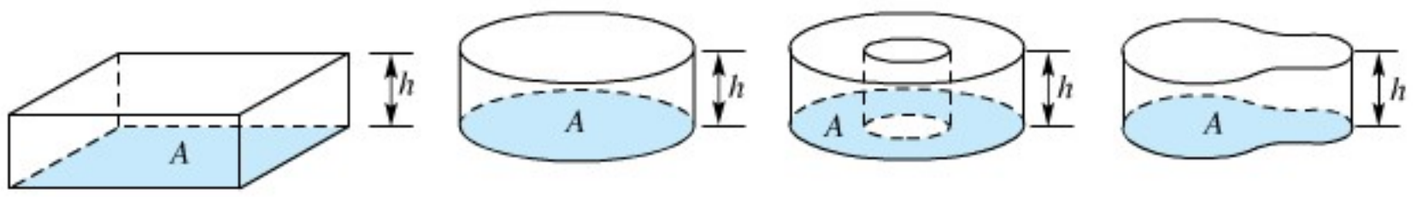


Figure 1

Next consider a solid with the property that cross sections perpendicular to a given line have known area. In particular, suppose that the line is the x-axis and that the area of the cross section at x is $A(x)$, $a \le x \le b$ (Figure 2). We partition the interval $[a, b]$ by inserting points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We then pass planes through these points perpendicular to the x-axis, thus slicing the solid into thin **slabs** (Figure 3). The *volume* ΔV_i of a slab should be approximately the volume of a cylinder; that is,

$$\Delta V_i \approx A(\overline{x}_i)\,\Delta x_i$$

The Volume of a Coin

Consider an ordinary coin, say a quarter.

A quarter has a radius of about 1 centimeter and a thickness of about 0.2 centimeter. Its volume is the area of the base, $A = \pi(1^2)$, times the thickness $h = 0.2$; that is,

$$V = (1\pi)(0.2) \approx 0.63$$

cubic centimeters.

(Recall that $\overline{x}_i$, called a *sample point*, is any number in the interval $[x_{i-1}, x_i]$.)

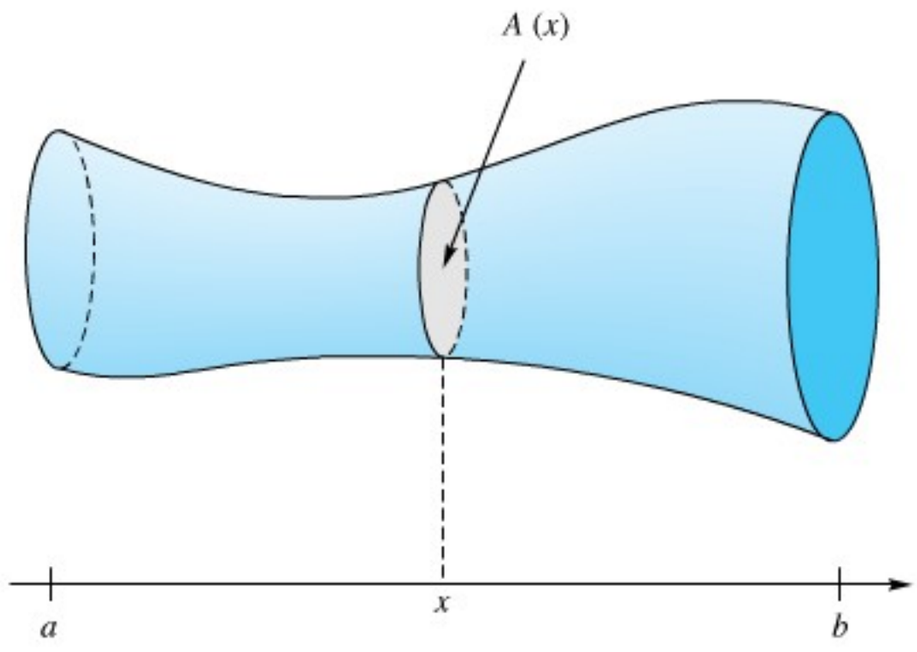


Figure 2

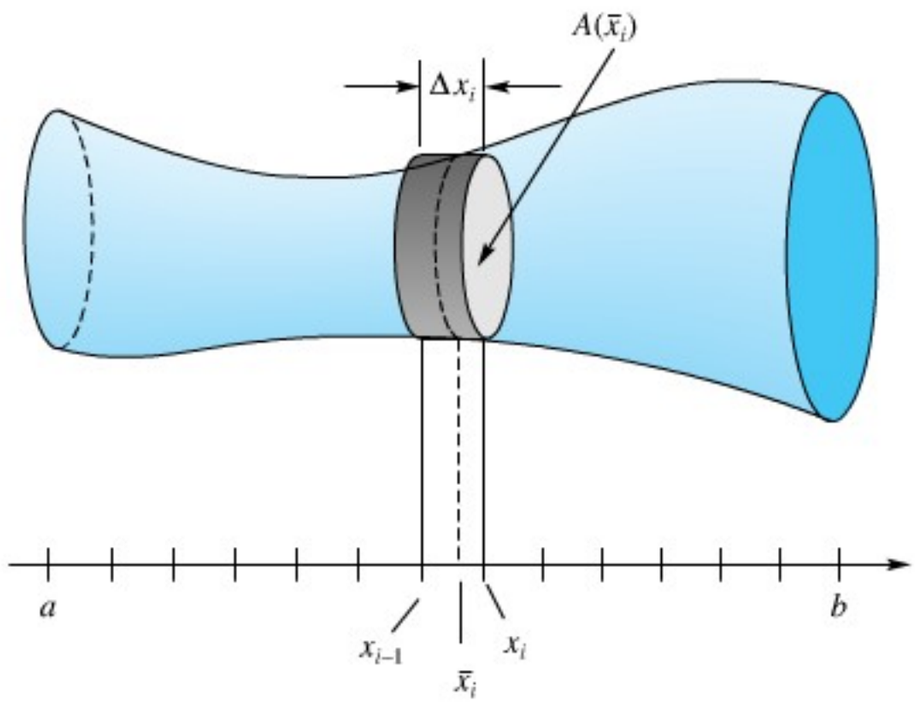


Figure 3

The "volume" V of the solid should be given approximately by the Riemann sum

$$V \approx \sum_{i=1}^{n} A(x_i)\,\Delta x_i$$

When we let the norm of the partition approach zero, we obtain a definite integral; this integral is defined to be the **volume** of the solid.

$$\boxed{V = \int_a^b A(x)\,dx}$$

Rather than routinely applying the boxed formula to obtain volumes, we suggest that in each problem you go through the process that led to it. Just as for areas, we call this process *slice, approximate, integrate*. It is illustrated in the examples that follow.

Solids of Revolution: Method of Disks When a plane region, lying entirely on one side of a fixed line in its plane, is revolved about that line, it generates a **solid of revolution.** The fixed line is called the **axis** of the solid of revolution.

As an illustration, if the region bounded by a semicircle and its diameter is revolved about that diameter, it sweeps out a spherical solid (Figure 4). If the region inside a right triangle is revolved about one of its legs, it generates a conical solid (Figure 5). When a circular region is revolved about a line in its plane that does not intersect the circle (Figure 6), it sweeps out a torus (doughnut). In each case, it is possible to represent the volume as a definite integral.

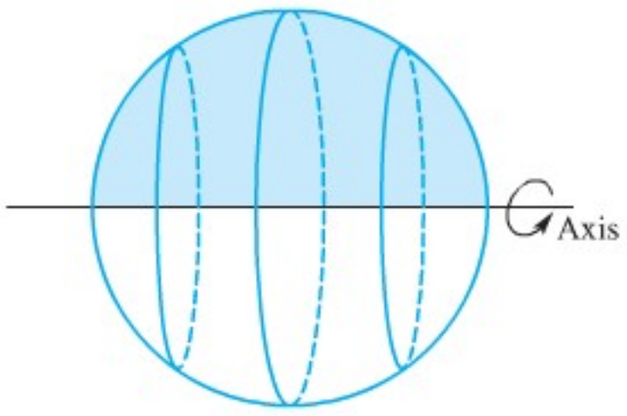


Figure 4

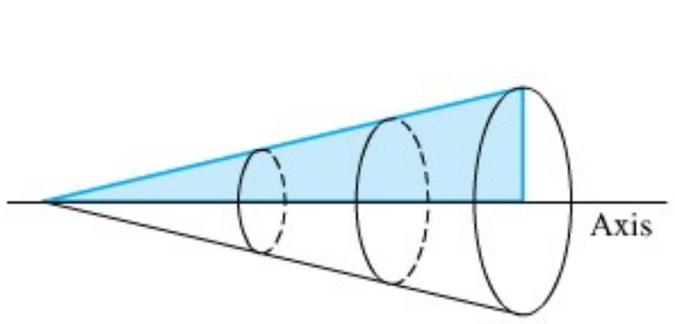


Figure 5

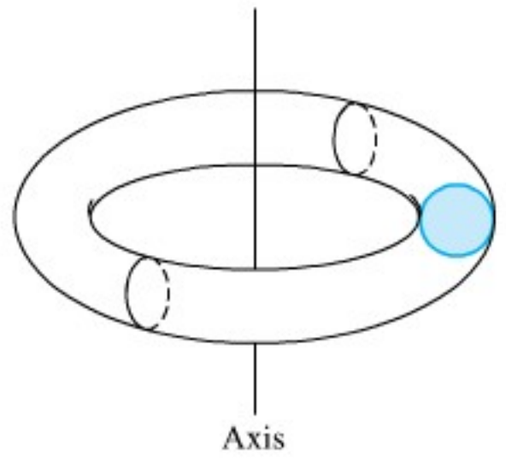


Figure 6

EXAMPLE 1 Find the volume of the solid of revolution obtained by revolving the plane region R bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$ about the x-axis.

SOLUTION The region R, with a typical slice, is displayed as the left part of Figure 7. When revolved about the x-axis, this region generates a solid of revolution and the slice generates a disk, a thin coin-shaped object.

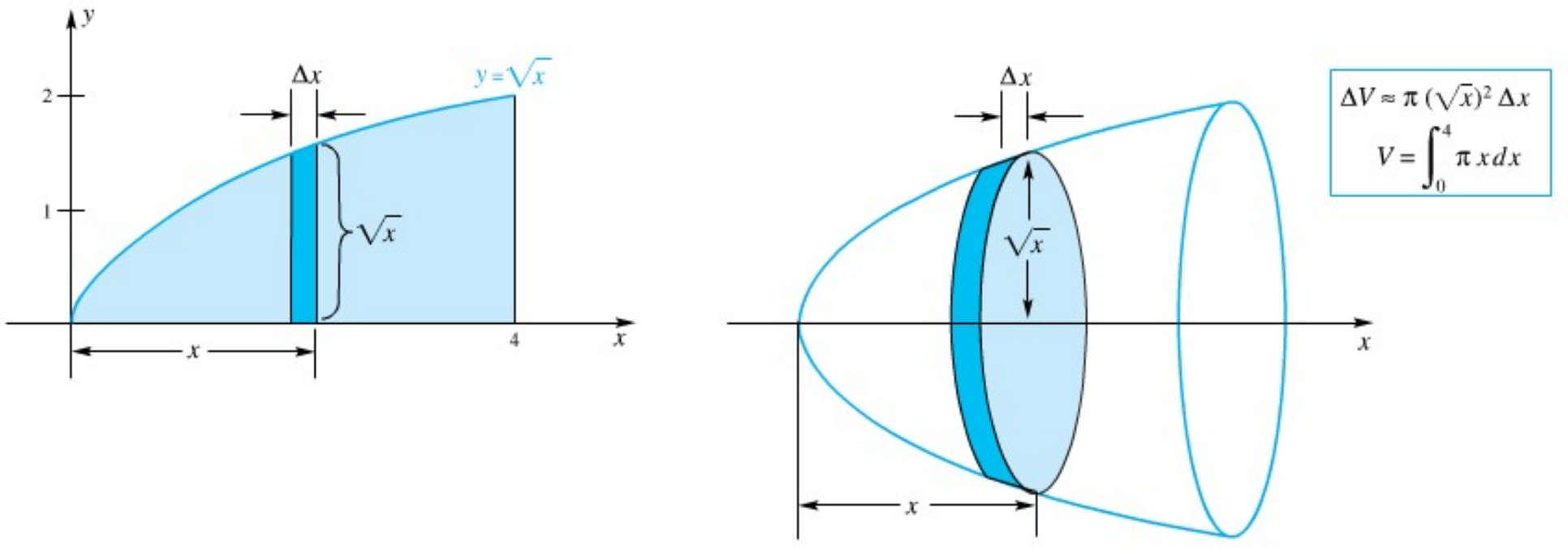


Figure 7

Recalling that the volume of a circular cylinder is $\pi r^2 h$, we approximate the volume ΔV of this disk with $\Delta V \approx \pi\left(\sqrt{x}\right)^2 \Delta x$ and then integrate.

$$V = \pi \int_0^4 x\, dx = \pi\left[\frac{x^2}{2}\right]_0^4 = \pi\frac{16}{2} = 8\pi \approx 25.13$$

≈ Is this answer reasonable? The right circular cylinder that contains the solid has volume $V = \pi 2^2 \cdot 4 = 16\pi$. Half this number seems reasonable. ■

EXAMPLE 2 Find the volume of the solid generated by revolving the region bounded by the curve $y = x^3$, the y-axis, and the line $y = 3$ about the y-axis (Figure 8).

SOLUTION Here we slice horizontally, which makes y the choice for the integration variable. Note that $y = x^3$ is equivalent to $x = \sqrt[3]{y}$ and $\Delta V \approx \pi\left(\sqrt[3]{y}\right)^2 \Delta y$.

The volume is therefore

$$V = \pi \int_0^3 y^{2/3}\, dy = \pi\left[\frac{3}{5} y^{5/3}\right]_0^3 = \pi\frac{9\sqrt[3]{9}}{5} \approx 11.76$$

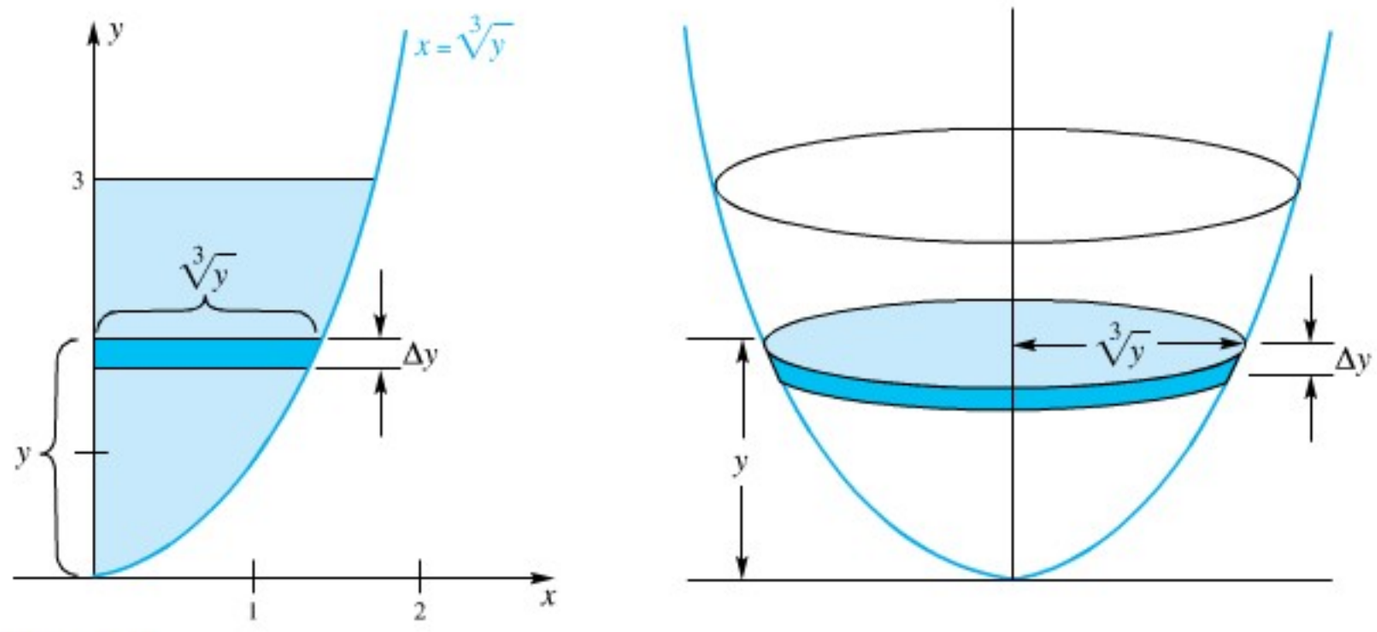


Figure 8

■

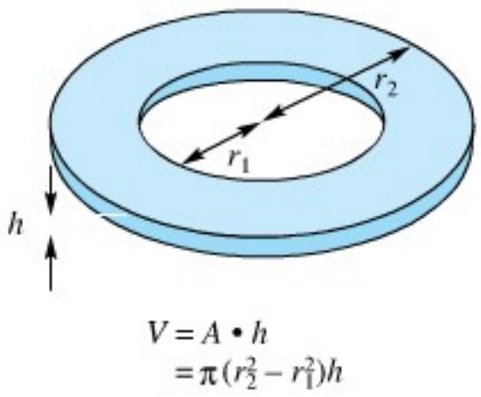


Figure 9

Method of Washers Sometimes, slicing a solid of revolution results in disks with holes in the middle. We call them **washers.** See the diagram and accompanying volume formula shown in Figure 9.

EXAMPLE 3 Find the volume of the solid generated by revolving the region bounded by the parabolas $y = x^2$ and $y^2 = 8x$ about the x-axis.

SOLUTION The key words are still *slice*, *approximate*, *integrate* (see Figure 10).

$$V = \pi \int_0^2 (8x - x^4)\,dx = \pi\left[\frac{8x^2}{2} - \frac{x^5}{5}\right]_0^2 = \frac{48\pi}{5} \approx 30.16$$

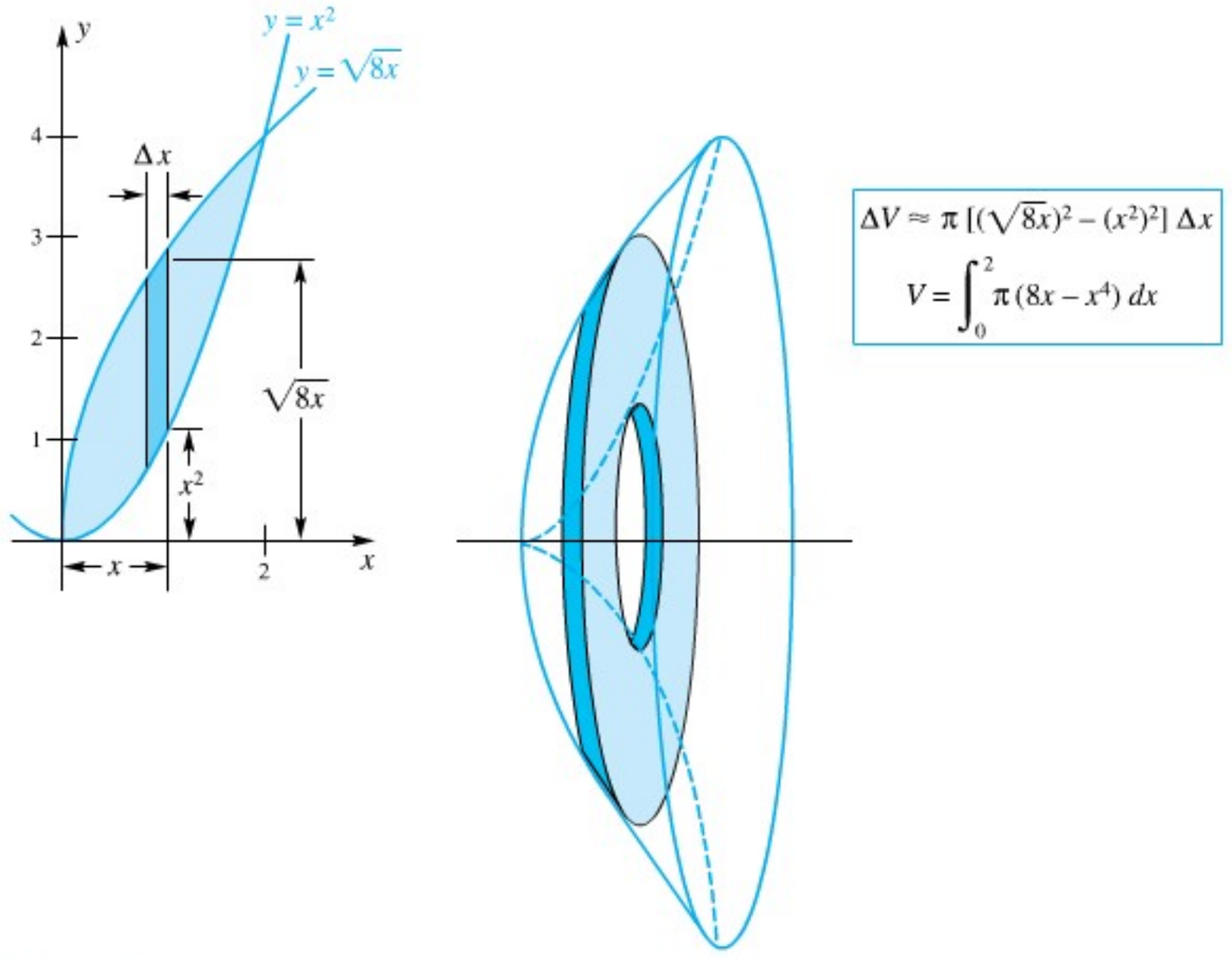


Figure 10 ■

EXAMPLE 4 The semicircular region bounded by the curve $x = \sqrt{4 - y^2}$ and the y-axis is revolved about the line $x = -1$. Set up the integral that represents its volume.

SOLUTION Here the outer radius of the washer is $1 + \sqrt{4 - y^2}$ and the inner radius is 1. Figure 11 exhibits the solution. The integral can be simplified. The part above the x-axis has the same volume as the part below it (which manifests itself in an even integrand). Thus, we may integrate from 0 to 2 and double the result.

$$V = \pi \int_{-2}^{2} \left[\left(1 + \sqrt{4 - y^2}\right)^2 - 1^2\right] dy$$

$$= 2\pi \int_0^2 \left[2\sqrt{4 - y^2} + 4 - y^2\right] dy$$

Now see Problem 35 for a way to evaluate this integral.

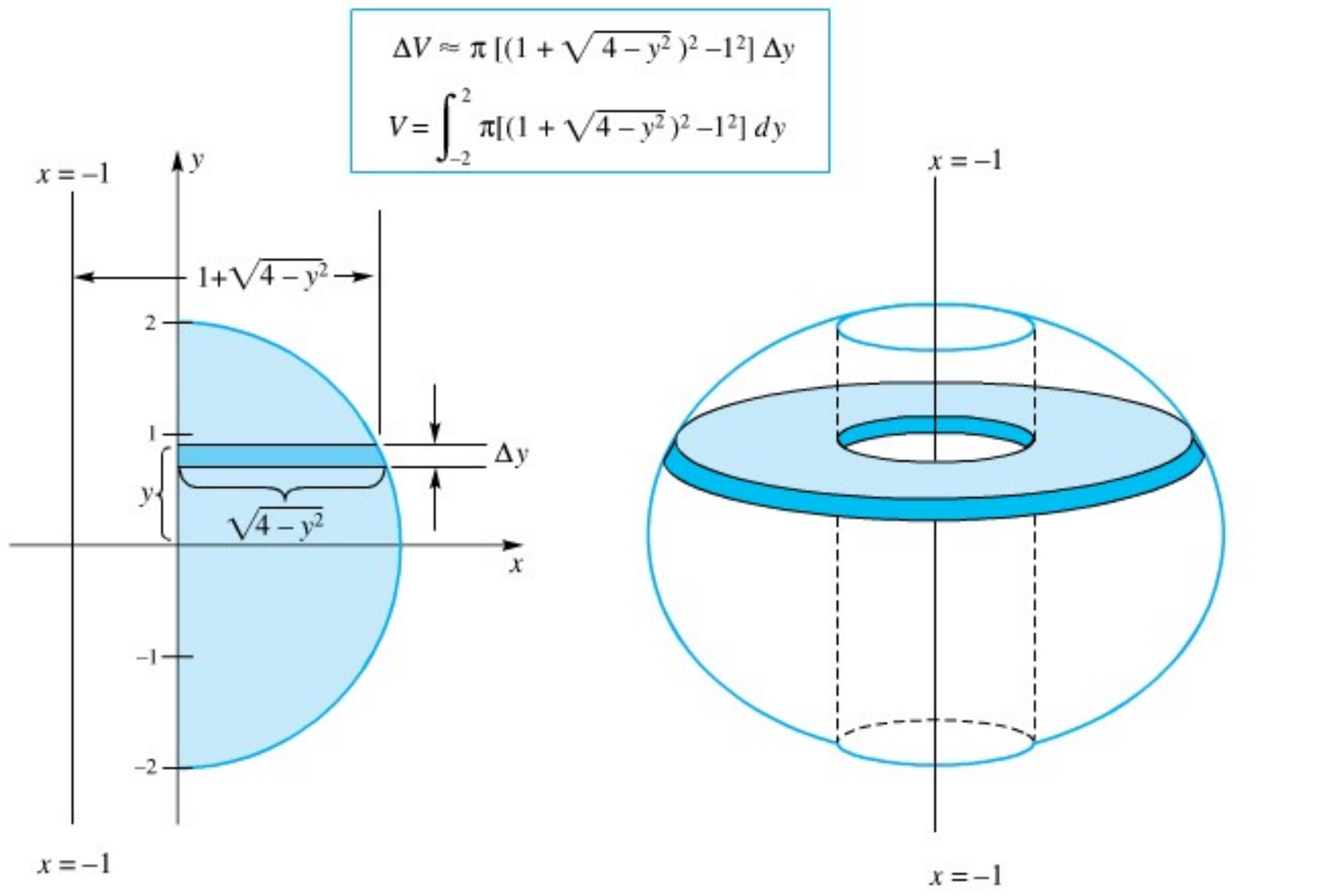


Figure 11 ■

Other Solids with Known Cross Sections So far, our solids have had circular cross sections. However, the method for finding volume works just as well for solids whose cross sections are squares or triangles. In fact, all that is really needed is that the areas of the cross sections can be determined, since, in this case, we can also approximate the volume of the slice—a slab—with this cross section. The volume is then found by integrating.

EXAMPLE 5 Let the base of a solid be the first quadrant plane region bounded by $y = 1 - x^2/4$, the x-axis, and the y-axis. Suppose that cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

SOLUTION When we slice this solid perpendicularly to the x-axis, we get thin square boxes (Figure 12), like slices of cheese.

$$V = \int_0^2 \left(1 - \frac{x^2}{4}\right)^2 dx = \int_0^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{16}\right) dx = \left[x - \frac{x^3}{6} + \frac{x^5}{80}\right]_0^2$$

$$= 2 - \frac{8}{6} + \frac{32}{80} = \frac{16}{15} \approx 1.07$$

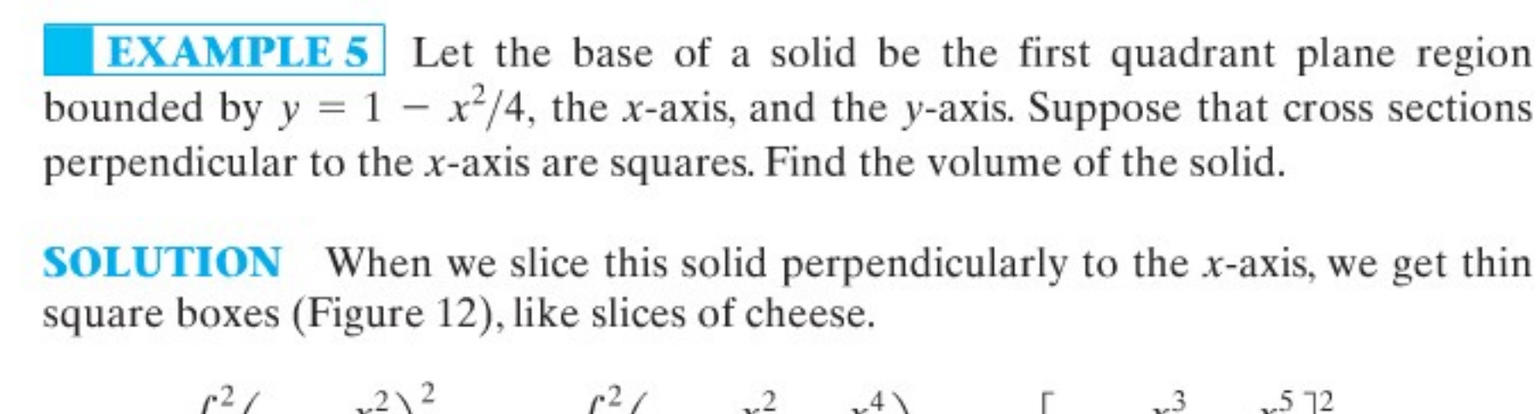
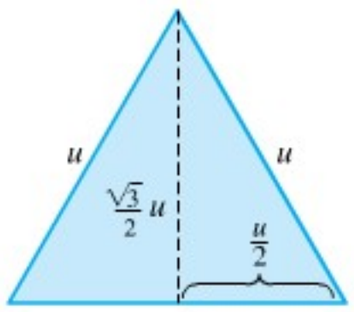


Figure 12 ■

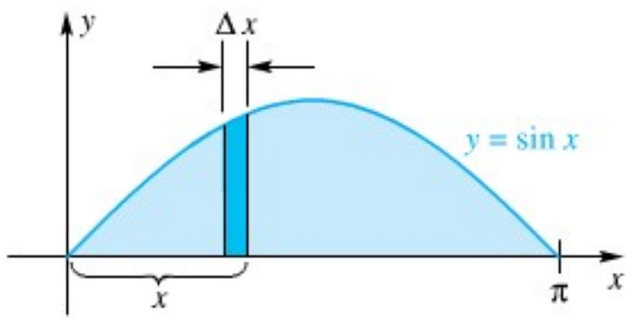


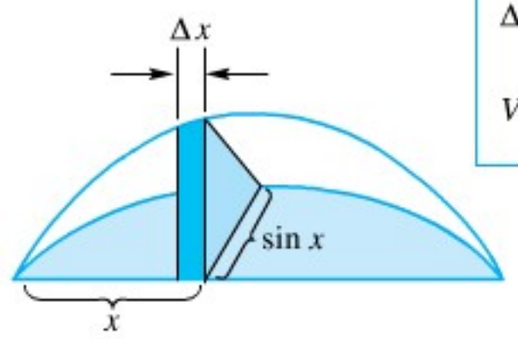
$$A = \frac{1}{2}u\left(\frac{\sqrt{3}}{2}u\right) = \frac{\sqrt{3}}{4}u^2$$

Figure 13

EXAMPLE 6 The base of a solid is the region between one arch of $y = \sin x$ and the x-axis. Each cross section perpendicular to the x-axis is an equilateral triangle sitting on this base. Find the volume of the solid.

SOLUTION We need the fact that the area of an equilateral triangle of side u is $\sqrt{3}\,u^2/4$ (see Figure 13). We proceed as shown in Figure 14.





$$\Delta V \approx \left(\frac{\sqrt{3}}{4}\sin^2 x\right)\Delta x$$

$$V = \int_0^{\pi}\left(\frac{\sqrt{3}}{4}\sin^2 x\right)dx$$

Figure 14

To perform the indicated integration, we use the half-angle formula $\sin^2 x = (1 - \cos 2x)/2$.

$$V = \frac{\sqrt{3}}{4}\int_0^{\pi}\frac{1-\cos 2x}{2}dx = \frac{\sqrt{3}}{8}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \frac{\sqrt{3}}{8}\left[\int_0^{\pi}1\,dx - \frac{1}{2}\int_0^{\pi}\cos 2x\cdot 2\,dx\right]$$

$$= \frac{\sqrt{3}}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\sqrt{3}}{8}\pi \approx 0.68$$

Concepts Review

1. The volume of a disk of radius r and thickness h is _____.

2. The volume of a washer of inner radius r, outer radius R, and thickness h is _____.

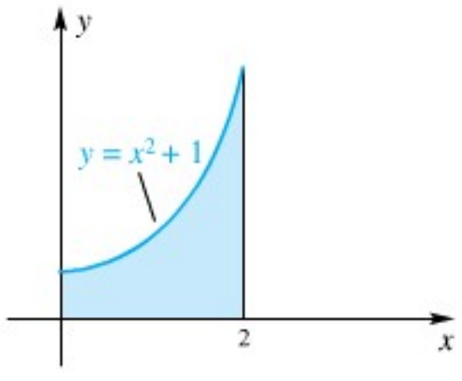
3. If the region R bounded by $y = x^2$, $y = 0$, and $x = 3$ is revolved about the x-axis, the disk at x will have volume $\Delta V \approx$ _____.

4. If the region R of Question 3 is revolved about the line $y = -2$, the washer at x will have volume $\Delta V \approx$ _____.

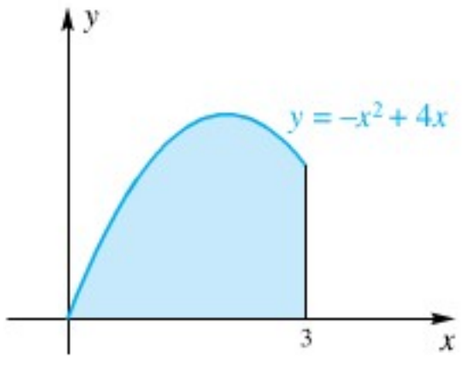
Problem Set 5.2

In Problems 1–4, find the volume of the solid generated when the indicated region is revolved about the specified axis; slice, approximate, integrate.

1. x-axis

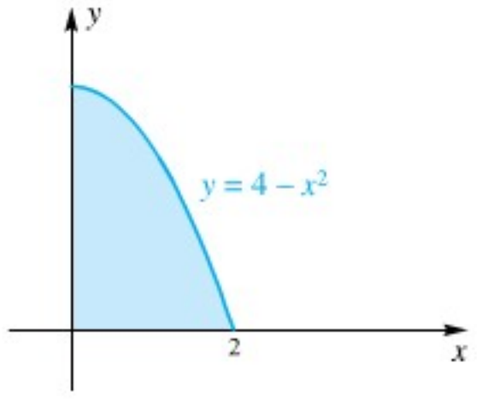


2. x-axis



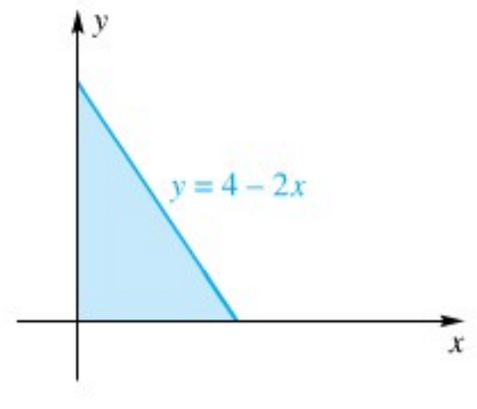
3. (a) x-axis

(b) y-axis



4. (a) x-axis

(b) y-axis



≈ *In Problems 5–10, sketch the region R bounded by the graphs of the given equations, and show a typical vertical slice. Then find the volume of the solid generated by revolving R about the x-axis.*

5. $y = \dfrac{x^2}{\pi}, x = 4, y = 0$

6. $y = x^3, x = 3, y = 0$

7. $y = \dfrac{1}{x}, x = 2, x = 4, y = 0$

8. $y = x^{3/2}, y = 0$, between $x = 2$ and $x = 3$

9. $y = \sqrt{9 - x^2}, y = 0$, between $x = -2$ and $x = 3$

10. $y = x^{2/3}, y = 0$, between $x = 1$ and $x = 27$

≈ *In Problems 11–16, sketch the region R bounded by the graphs of the given equations and show a typical horizontal slice. Find the volume of the solid generated by revolving R about the y-axis.*

11. $x = y^2, x = 0, y = 3$

12. $x = \dfrac{2}{y}, y = 2, y = 6, x = 0$

13. $x = 2\sqrt{y}, y = 4, x = 0$ **14.** $x = y^{2/3}, y = 27, x = 0$

15. $x = y^{3/2}, y = 9, x = 0$ **16.** $x = \sqrt{4 - y^2}, x = 0$

17. Find the volume of the solid generated by revolving about the x-axis the region bounded by the upper half of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the x-axis, and thus find the volume of a *prolate spheroid*. Here a and b are positive constants, with $a > b$.

18. Find the volume of the solid generated by revolving about the x-axis the region bounded by the line $y = 6x$ and the parabola $y = 6x^2$.

19. Find the volume of the solid generated by revolving about the x-axis the region bounded by the line $x - 2y = 0$ and the parabola $y^2 = 4x$.

20. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant bounded by the circle $x^2 + y^2 = r^2$, the x-axis, and the line $x = r - h, 0 < h < r$, and thus find the volume of a *spherical segment* of height h, of a sphere of radius r.

21. Find the volume of the solid generated by revolving about the y-axis the region bounded by the line $y = 4x$ and the parabola $y = 4x^2$.

22. Find the volume of the solid generated by revolving about the line $y = 2$ the region in the first quadrant bounded by the parabolas $3x^2 - 16y + 48 = 0$ and $x^2 - 16y + 80 = 0$ and the y-axis.

23. The base of a solid is the region inside the circle $x^2 + y^2 = 4$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a square. *Hint:* See Examples 5 and 6.

24. Do Problem 23 assuming that every cross section by a plane perpendicular to the x-axis is an isosceles triangle with base on the xy-plane and altitude 4. *Hint:* To complete the evaluation, interpret $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ as the area of a semicircle.

25. The base of a solid is bounded by one arch of $y = \sqrt{\cos x}, -\pi/2 \le x \le \pi/2$, and the x-axis. Each cross section perpendicular to the x-axis is a square sitting on this base. Find the volume of the solid.

26. The base of a solid is the region bounded by $y = 1 - x^2$ and $y = 1 - x^4$. Cross sections of the solid that are perpendicular to the x-axis are squares. Find the volume of the solid.

27. Find the volume of one octant (one-eighth) of the solid region common to two right circular cylinders of radius 1 whose axes intersect at right angles. *Hint:* Horizontal cross sections are squares. See Figure 15.

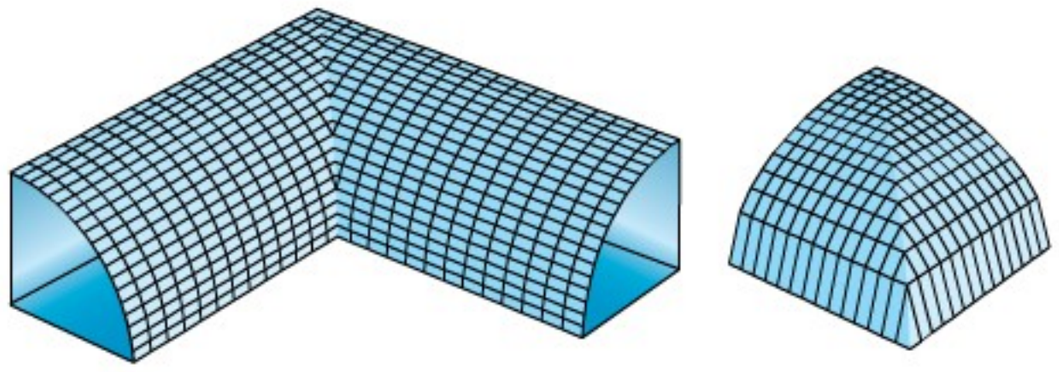


Figure 15

28. Find the volume inside the "+" shown in Figure 16. Assume that both cylinders have radius 2 inches and length 12 inches. *Hint:* The volume is equal to the volume of the first cylinder plus the volume of the second cylinder minus the volume of the region common to both. Use the result of Problem 27.

29. Find the volume inside the "+" in Figure 16, assuming that both cylinders have radius r and length L.

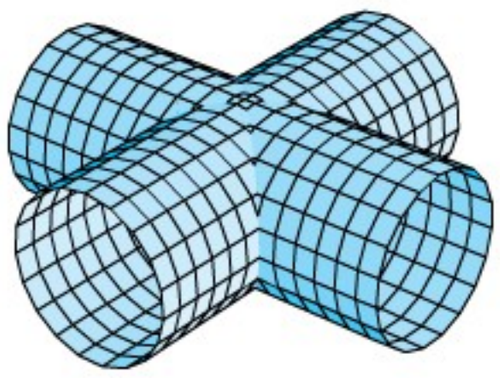


Figure 16

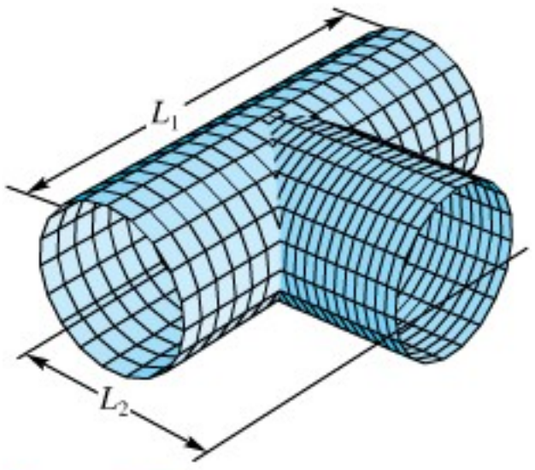


Figure 17

30. Find the volume inside the "T" in Figure 17, assuming that each cylinder has radius $r = 2$ inches and that the lengths are $L_1 = 12$ inches and $L_2 = 8$ inches.

31. Repeat Problem 30 for arbitrary r, L_1, and L_2.

32. The base of a solid is the region R bounded by $y = \sqrt{x}$ and $y = x^2$. Each cross section perpendicular to the x-axis is a semicircle with diameter extending across R. Find the volume of the solid.

33. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the curve $y^2 = x^3$, the line $x = 4$, and the x-axis:

(a) about the line $x = 4$; (b) about the line $y = 8$.

34. Find the volume of the solid generated by revolving the region bounded by the curve $y^2 = x^3$, the line $y = 8$, and the y-axis:

(a) about the line $x = 4$; (b) about the line $y = 8$.

35. Complete the evaluation of the integral in Example 4 by noting that

$$\int_0^2 \left[2\sqrt{4 - y^2} + 4 - y^2\right] dy$$

$$= 2\int_0^2 \sqrt{4 - y^2}\, dy + \int_0^2 (4 - y^2)\, dy$$

Now interpret the first integral as the area of a quarter circle.

36. An open barrel of radius r and height h is initially full of water. It is tilted and water pours out until the water level coincides with a diameter of the base and just touches the rim of the top. Find the volume of water left in the barrel. See Figure 18.

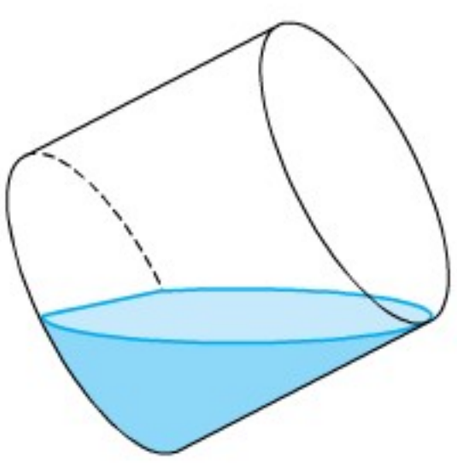


Figure 18

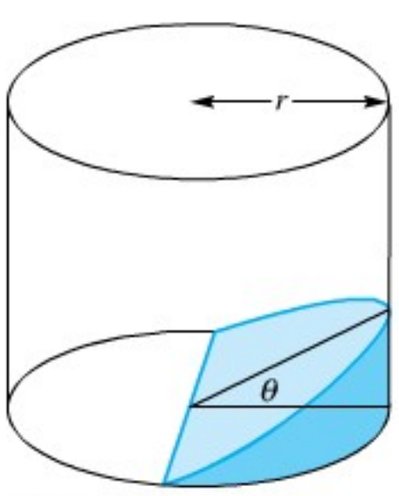


Figure 19

37. A wedge is cut from a right circular cylinder of radius r (Figure 19). The upper surface of the wedge is in a plane through a diameter of the circular base and makes an angle θ with the base. Find the volume of the wedge.

38. (The Water Clock) A water tank is obtained by revolving the curve $y = kx^4$, $k > 0$, about the y-axis.

(a) Find $V(y)$, the volume of water in the tank as a function of its depth y.

(b) Water drains through a small hole according to Torricelli's Law $\left(dV/dt = -m\sqrt{y}\right)$. Show that the water level falls at a constant rate.

39. Show that the volume of a general cone (Figure 20) is $\frac{1}{3}Ah$, where A is the area of the base and h is the height. Use this result to give the formula for the volume of

(a) a right circular cone of radius r and height h;

(b) a regular tetrahedron with edge length r.

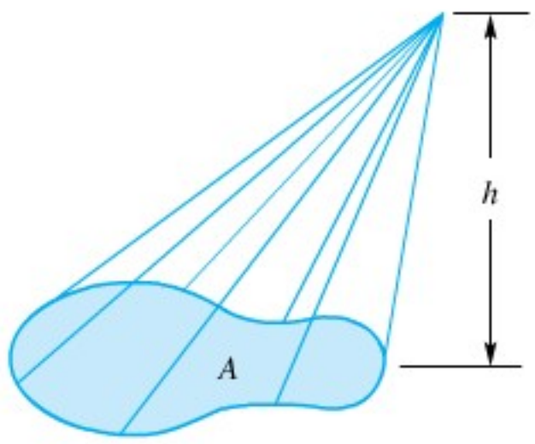


Figure 20

40. State the version of Cavalieri's Principle for volume (see Problem 36 of Section 5.1).

41. Apply Cavalieri's Principle for volumes to the two solids shown in Figure 21. (One is a hemisphere of radius r; the other is a cylinder of radius r and height r with a right circular cone of radius r and height r removed.) Assuming that the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, find the volume of a hemisphere of radius r.

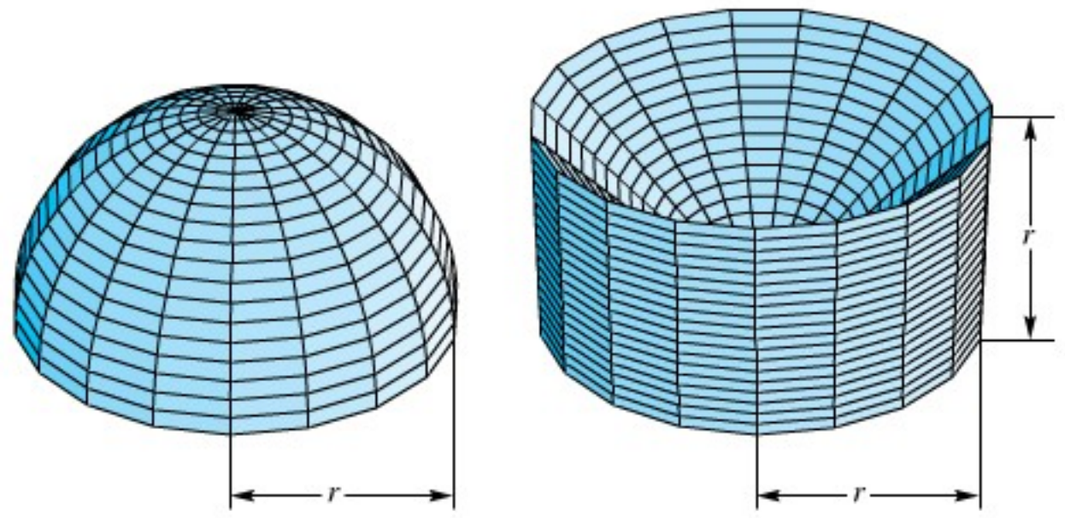


Figure 21

Answers to Concepts Review: **1.** $\pi r^2 h$ **2.** $\pi(R^2 - r^2)h$ **3.** $\pi x^4\, \Delta x$ **4.** $\pi[(x^2 + 2)^2 - 4]\, \Delta x$

5.3 Volumes of Solids of Revolution: Shells

There is another method for finding the volume of a solid of revolution: the method of cylindrical shells. For many problems, it is easier to apply than the methods of disks or washers.

A cylindrical shell is a solid bounded by two concentric right circular cylinders (Figure 1). If the inner radius is r_1, the outer radius is r_2, and the height is h, then its volume is given by

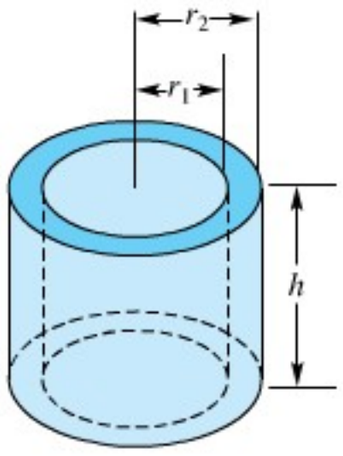


Figure 1

$$\begin{aligned} V &= (\text{area of base}) \cdot (\text{height}) \\ &= (\pi r_2^2 - \pi r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi\left(\frac{r_2 + r_1}{2}\right)h(r_2 - r_1) \end{aligned}$$

The expression $(r_1 + r_2)/2$, which we will denote by r, is the average of r_1 and r_2. Thus,

$$\begin{aligned} V &= 2\pi \cdot (\text{average radius}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= 2\pi r h\, \Delta r \end{aligned}$$

Here is a good way to remember this formula: If the shell were very thin and flexible (like paper), we could slit it down the side, open it up to form a rectangular sheet, and then calculate its volume by pretending that this sheet forms a thin box of length $2\pi r$, height h, and thickness Δr (Figure 2).

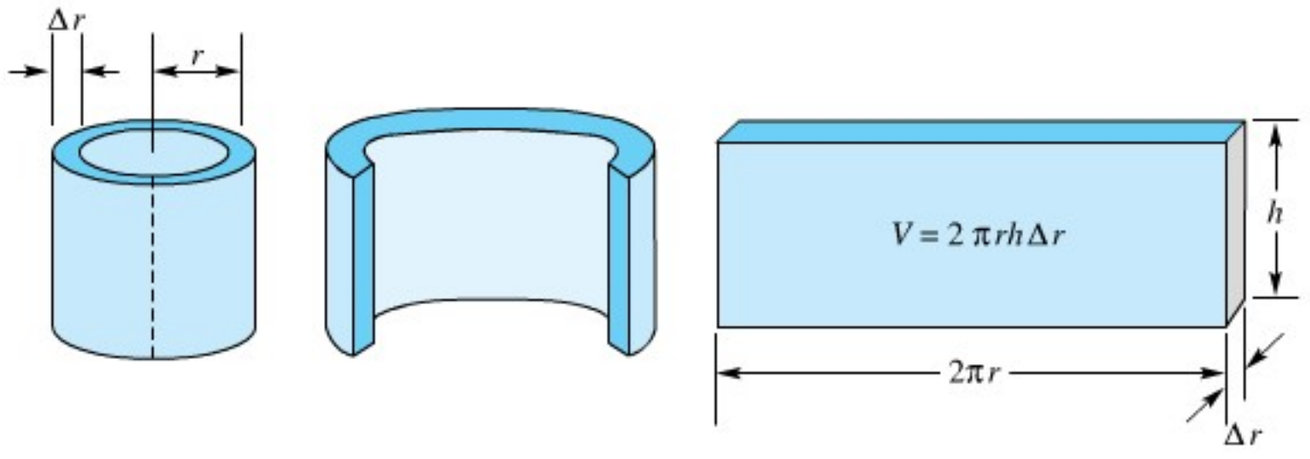


Figure 2

The Method of Shells Consider now a region of the type shown in Figure 3. Slice it vertically and revolve it about the y-axis. It will generate a solid of revolution, and each slice will generate a piece that is approximately a cylindrical shell. To get the volume of this solid, we calculate the volume ΔV of a typical shell, add, and take the limit as the thickness of the shells tends to zero. The latter is, of course, an integral. Again, the strategy is *slice*, *approximate*, *integrate*.

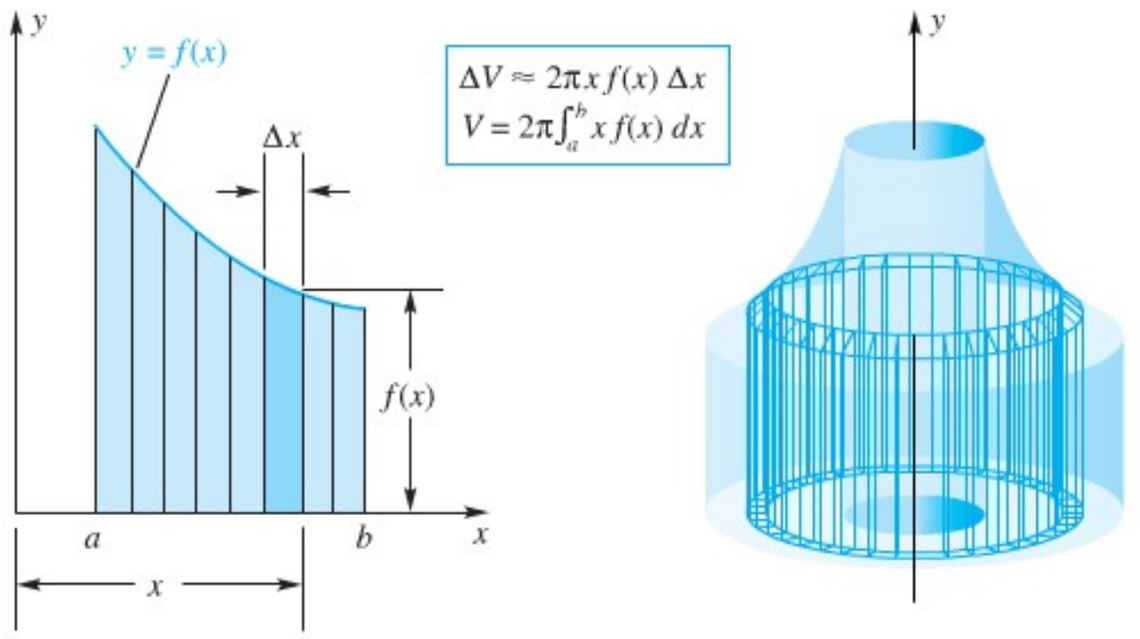


Figure 3

EXAMPLE 1 The region bounded by $y = 1/\sqrt{x}$, the x-axis, $x = 1$, and $x = 4$ is revolved about the y-axis. Find the volume of the resulting solid.

SOLUTION From Figure 3 we see that the volume of the shell generated by the slice is

$$\Delta V \approx 2\pi x f(x)\, \Delta x$$

which, for $f(x) = 1/\sqrt{x}$, becomes

$$\Delta V \approx 2\pi x \frac{1}{\sqrt{x}} \Delta x$$

The volume is then found by integrating.

$$V = 2\pi \int_1^4 x \frac{1}{\sqrt{x}}\, dx = 2\pi \int_1^4 x^{1/2}\, dx$$

$$= 2\pi \left[\frac{2}{3}x^{3/2}\right]_1^4 = 2\pi\left(\frac{2}{3}\cdot 8 - \frac{2}{3}\cdot 1\right) = \frac{28\pi}{3} \approx 29.32$$ ■

EXAMPLE 2 The region bounded by the line $y = (r/h)x$, the x-axis, and $x = h$ is revolved about the x-axis, thereby generating a cone (assume that $r > 0, h > 0$). Find its volume by the disk method and by the shell method.

SOLUTION

Disk Method Follow the steps suggested by Figure 4; that is, *slice*, *approximate*, *integrate*.

$$V = \pi \frac{r^2}{h^2} \int_0^h x^2\, dx = \pi \frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h^3}{3h^2} = \frac{1}{3}\pi r^2 h$$

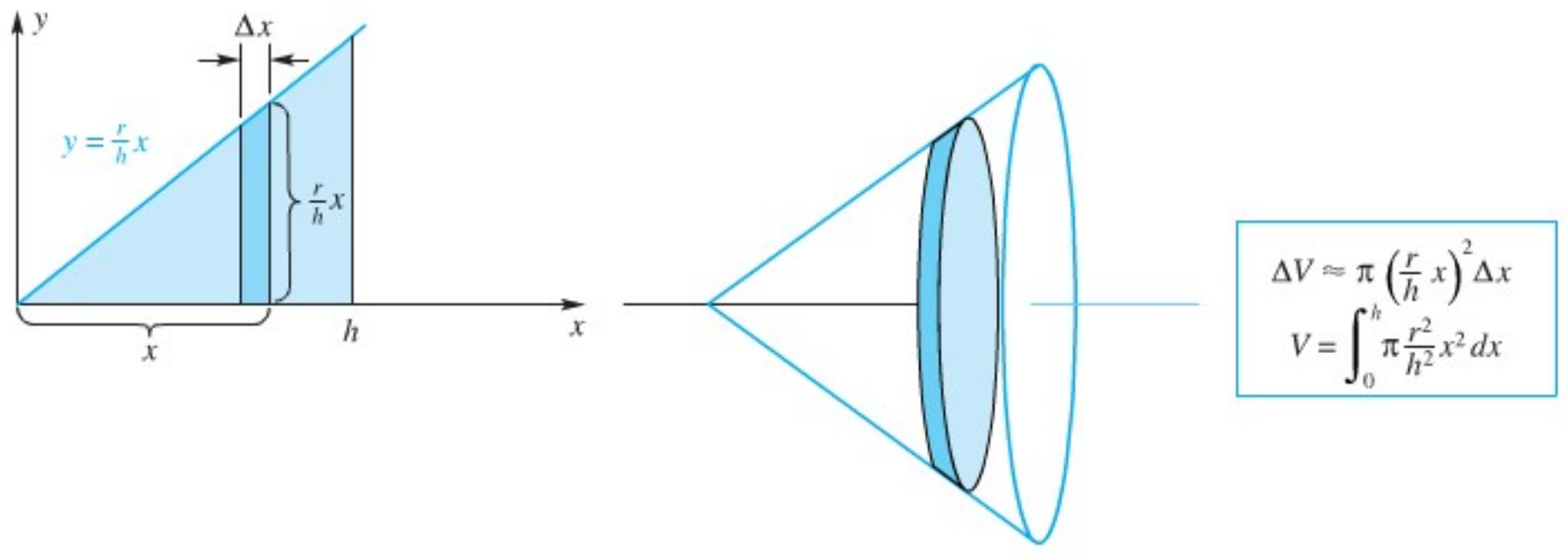


Figure 4

Shell Method Follow the steps suggested by Figure 5. The volume is then

$$V = \int_0^r 2\pi y\left(h - \frac{h}{r}y\right) dy = 2\pi h \int_0^r \left(y - \frac{1}{r}y^2\right) dy$$

$$= 2\pi h\left[\frac{y^2}{2} - \frac{y^3}{3r}\right]_0^r = 2\pi h\left[\frac{r^2}{2} - \frac{r^2}{3}\right] = \frac{1}{3}\pi r^2 h$$

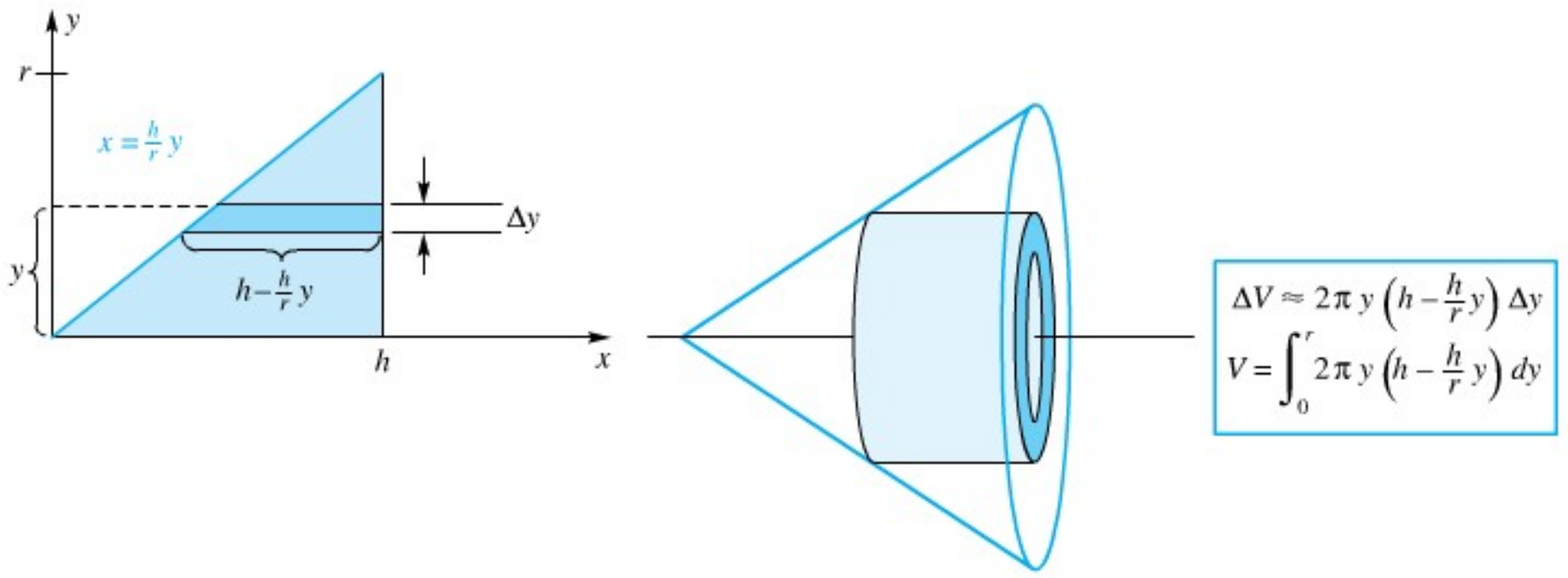


Figure 5

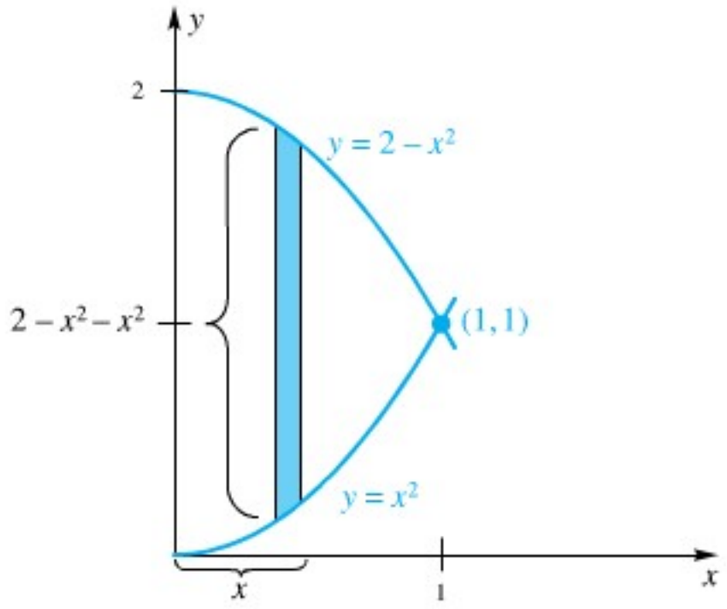
As should be expected, both methods yield the well-known formula for the volume of a right circular cone. ■

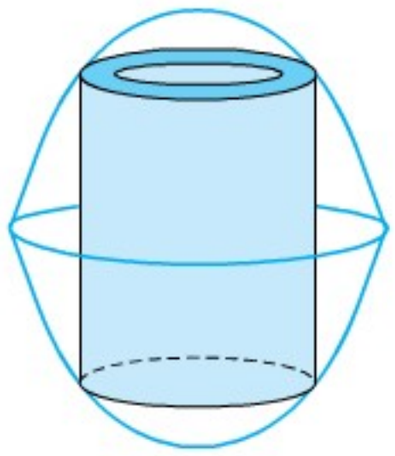
EXAMPLE 3 Find the volume of the solid generated by revolving the region in the first quadrant that is above the parabola $y = x^2$ and below the parabola $y = 2 - x^2$ about the y-axis.

SOLUTION One look at the region (left part of Figure 6) should convince you that horizontal slices leading to the disk method are not the best choice (because the right boundary consists of parts of two curves, making it necessary to use two integrals). However, vertical slices, resulting in cylindrical shells, will work fine.

$$V = \int_0^1 2\pi x(2 - 2x^2)\,dx = 4\pi \int_0^1 (x - x^3)\,dx$$

$$= 4\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 4\pi\left[\frac{1}{2} - \frac{1}{4}\right] = \pi \approx 3.14$$





$$\Delta V \approx 2\pi x(2 - x^2 - x^2)\,\Delta x$$

$$V = \int_0^1 2\pi x\,(2 - 2x^2)\,dx$$

Figure 6

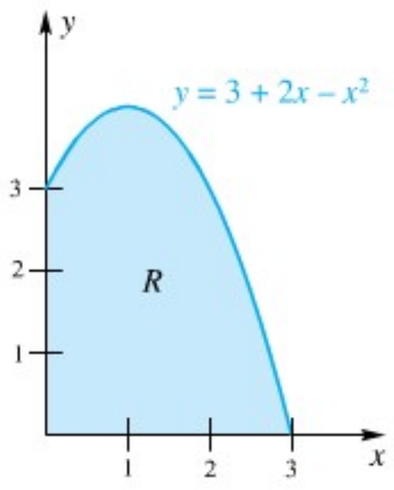


Figure 7

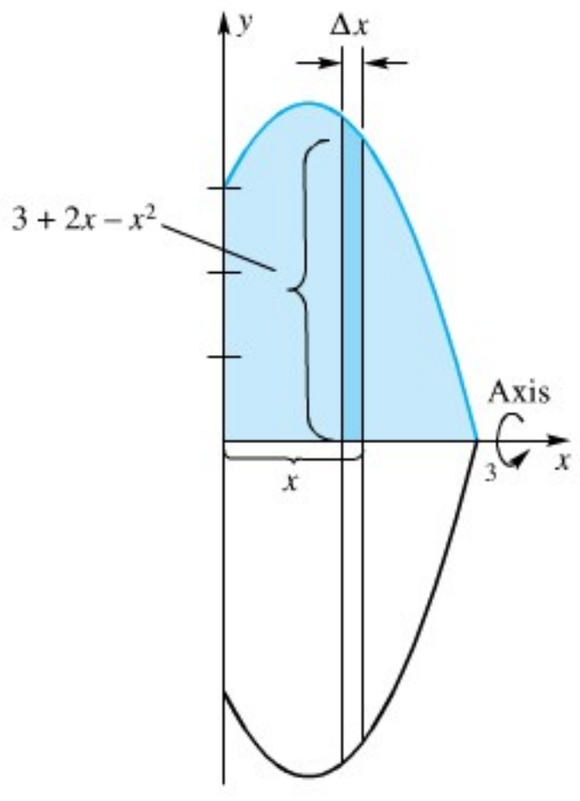
Putting It All Together Although most of us can draw a reasonably good plane figure, some of us do less well at drawing three-dimensional solids. But no law says that we have to draw a solid in order to calculate its volume. Usually, a plane figure will do, provided we can visualize the corresponding solid in our minds. In the next example, we are going to imagine revolving the region R of Figure 7 about various axes. Our job is to set up and evaluate an integral for the volume of the resulting solid, and we are going to do it by looking at a plane figure.

EXAMPLE 4 Set up and evaluate an integral for the volume of the solid that results when the region R shown in Figure 7 is revolved about

(a) the x-axis, (b) the y-axis,
(c) the line $y = -1$, (d) the line $x = 4$.

SOLUTION

(a)



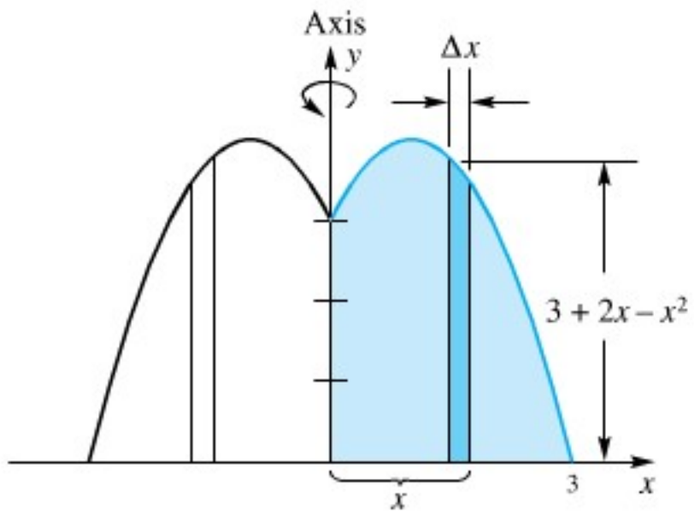
Method of disks

$$\Delta V \approx \pi(3 + 2x - x^2)^2\,\Delta x$$

$$V = \pi \int_0^3 (3 + 2x - x^2)^2\,dx$$

$$V = \pi \int_0^3 (3 + 2x - x^2)^2\,dx = \frac{153}{5}\pi \approx 96.13$$

(b)



Method of shells

$$\Delta V \approx 2\pi x\,(3 + 2x - x^2)\,\Delta x$$

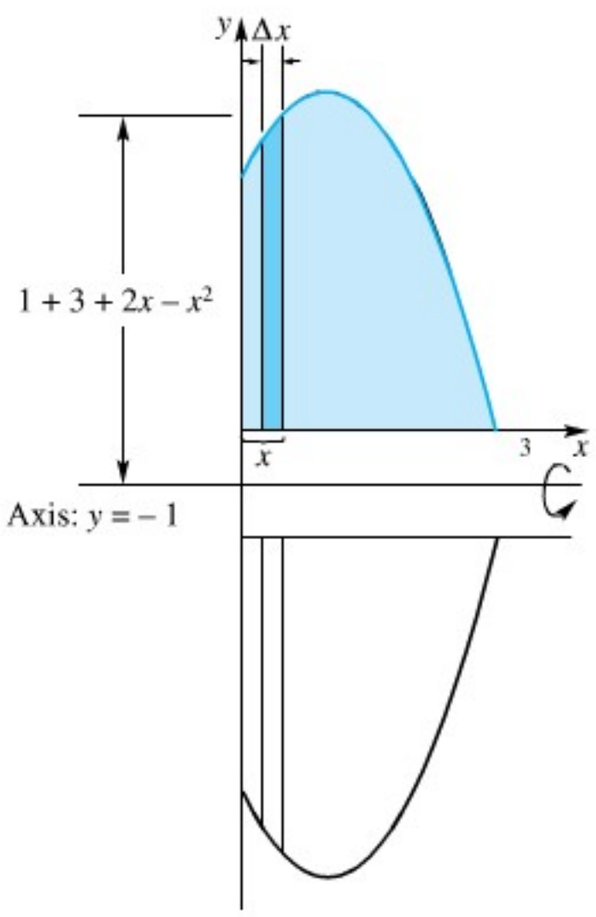
$$V = 2\pi \int_0^3 x\,(3 + 2x - x^2)\,dx$$

$$V = 2\pi \int_0^3 x(3 + 2x - x^2)\,dx = \frac{45}{2}\pi \approx 70.69$$

Technology

In all four parts of this example, the integrand turned out to be a polynomial, but finding the polynomial involved some messy expansions. Once the integrals are set up, evaluating them is an ideal task for a CAS.

(c)



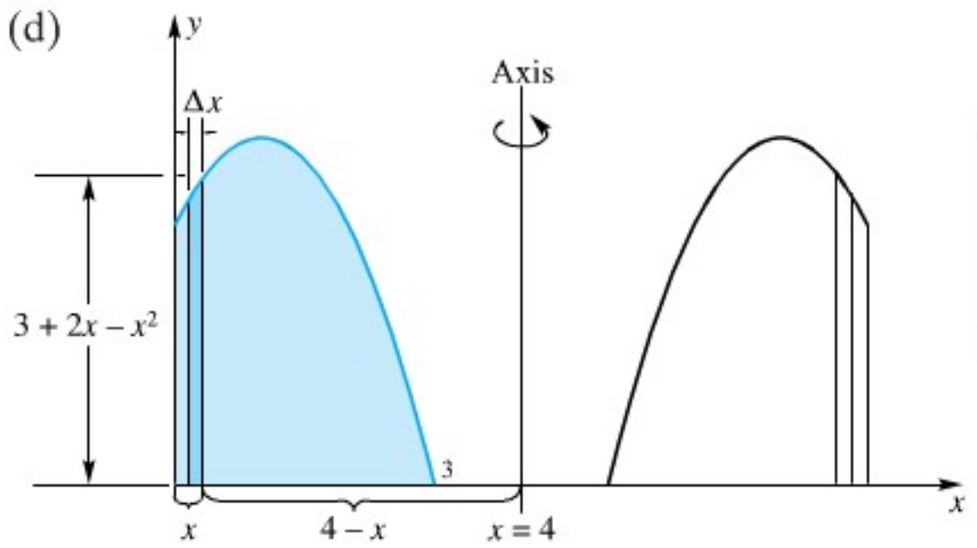
Method of washers

$$\Delta V \approx \pi[(4 + 2x - x^2)^2 - 1^2]\,\Delta x$$

$$V = \pi \int_0^3 [(4 + 2x - x^2)^2 - 1]\,dx$$

$$V = \pi \int_0^3 [(4 + 2x - x^2)^2 - 1]\,dx = \frac{243}{5}\pi \approx 152.68$$

(d)



Method of shells

$$\Delta V \approx 2\pi\,(4 - x)(3 + 2x - x^2)\,\Delta x$$

$$V = 2\pi \int_0^3 (4 - x)(3 + 2x - x^2)\,dx$$

$$V = 2\pi \int_0^3 (4 - x)(3 + 2x - x^2)\,dx = \frac{99}{2}\pi \approx 155.51$$

Note that in all four cases the limits of integration are the same; it is the original plane region that determines these limits. ■

Concepts Review

1. The volume ΔV of a thin cylindrical shell of radius x, height $f(x)$, and thickness Δx is given by $\Delta V \approx$ _____.

2. The triangular region R bounded by $y = x$, $y = 0$, and $x = 2$ is revolved about the y-axis, generating a solid. The method of shells gives the integral _____ as its volume; the method of washers gives the integral _____ as its volume.

3. The region R of Question 2 is revolved about the line $x = -1$, generating a solid. The method of shells gives the integral _____ as its volume.

4. The region R of Question 2 is revolved about the line $y = -1$, generating a solid. The method of shells gives the integral _____ as its volume.

Problem Set 5.3

In Problems 1–12, find the volume of the solid generated when the region R bounded by the given curves is revolved about the indicated axis. Do this by performing the following steps.

(a) Sketch the region R.
(b) Show a typical rectangular slice properly labeled.
(c) Write a formula for the approximate volume of the shell generated by this slice.
(d) Set up the corresponding integral.
(e) Evaluate this integral.

1. $y = \dfrac{1}{x}$, $x = 1$, $x = 4$, $y = 0$; about the y-axis

2. $y = x^2$, $x = 1$, $y = 0$; about the y-axis

3. $y = \sqrt{x}$, $x = 3$, $y = 0$; about the y-axis

4. $y = 9 - x^2 (x \geq 0)$, $x = 0$, $y = 0$; about the y-axis

5. $y = \sqrt{x}$, $x = 5$, $y = 0$; about the line $x = 5$

6. $y = 9 - x^2 (x \geq 0)$, $x = 0$, $y = 0$; about the line $x = 3$

7. $y = \frac{1}{4}x^3 + 1$, $y = 1 - x$, $x = 1$; about the y-axis

8. $y = x^2$, $y = 3x$; about the y-axis

9. $x = y^2$, $y = 1$, $x = 0$; about the x-axis

10. $x = \sqrt{y} + 1$, $y = 4$, $x = 0$, $y = 0$; about the x-axis

11. $x = y^2$, $y = 2$, $x = 0$; about the line $y = 2$

12. $x = \sqrt{2y} + 1$, $y = 2$, $x = 0$, $y = 0$; about the line $y = 3$

13. Consider the region R (Figure 8). Set up an integral for the volume of the solid obtained when R is revolved about the given line using the indicated method.

(a) The x-axis (washers) (b) The y-axis (shells)
(c) The line $x = a$ (shells) (d) The line $x = b$ (shells)

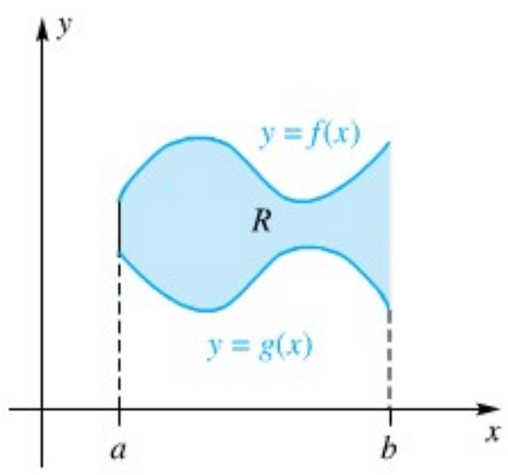


Figure 8

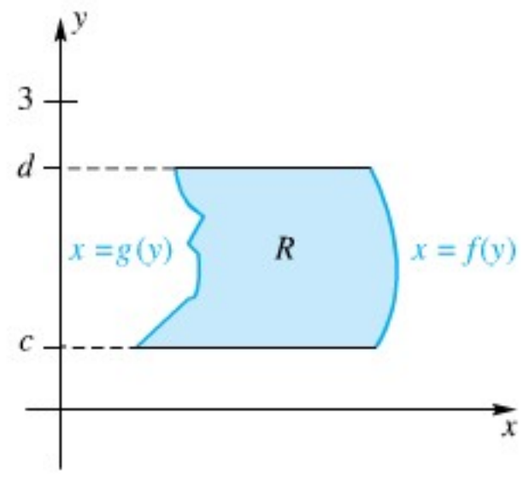


Figure 9

14. A region R is shown in Figure 9. Set up an integral for the volume of the solid obtained when R is revolved about each of the following lines. Use the indicated method.

(a) The x-axis (washers) (b) The y-axis (shells)
(c) The line $y = 3$ (shells)

15. Sketch the region R bounded by $y = 1/x^3$, $x = 1$, $x = 3$, and $y = 0$. Set up (but do not evaluate) integrals for each of the following.

(a) Area of R
(b) Volume of the solid obtained when R is revolved about the y-axis
(c) Volume of the solid obtained when R is revolved about $y = -1$
(d) Volume of the solid obtained when R is revolved about $x = 4$

16. Follow the directions of Problem 15 for the region R bounded by $y = x^3 + 1$ and $y = 0$ and between $x = 0$ and $x = 2$.

17. Find the volume of the solid generated by revolving the region R bounded by the curves $x = \sqrt{y}$ and $x = y^3/32$ about the x-axis.

18. Follow the directions of Problem 17, but revolve R about the line $y = 4$.

19. A round hole of radius a is drilled through the center of a solid sphere of radius b (assume that $b > a$). Find the volume of the solid that remains.

20. Set up the integral (using shells) for the volume of the torus obtained by revolving the region inside the circle $x^2 + y^2 = a^2$ about the line $x = b$, where $b > a$. Then evaluate this integral. *Hint:* As you simplify, it may help to think of part of this integral as an area.

21. The region in the first quadrant bounded by $x = 0$, $y = \sin(x^2)$, and $y = \cos(x^2)$ is revolved about the y-axis. Find the volume of the resulting solid.

22. The region bounded by $y = 2 + \sin x$, $y = 0$, $x = 0$, and $x = 2\pi$ is revolved about the y-axis. Find the volume that results.

Hint: $\displaystyle\int x \sin x \, dx = \sin x - x \cos x + C$.

23. Let R be the region bounded by $y = x^2$ and $y = x$. Find the volume of the solid that results when R is revolved around:

(a) the x-axis; (b) the y-axis; (c) the line $y = x$.

24. Suppose that we know the formula $S = 4\pi r^2$ for the surface area of a sphere, but do not know the corresponding formula for its volume V. Obtain this formula by slicing the solid sphere into thin concentric *spherical shells* (Figure 10). *Hint:* The volume ΔV of a thin spherical shell of outer radius x is $\Delta V \approx 4\pi x^2 \, \Delta x$.

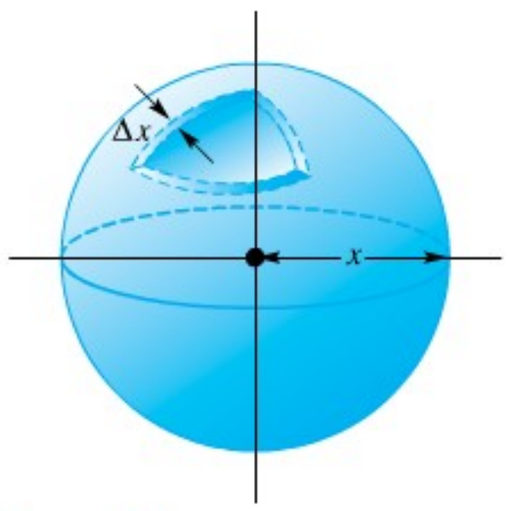


Figure 10

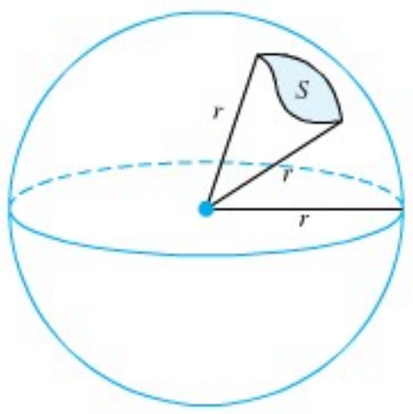


Figure 11

25. Consider a region of area S on the surface of a sphere of radius r. Find the volume of the solid that results when each point of this region is connected to the center of the sphere by a line segment (Figure 11). *Hint:* Use the method of spherical shells mentioned in Problem 24.

Answers to Concepts Review: **1.** $2\pi x f(x)\,\Delta x$ **2.** $2\pi \int_0^2 x^2\,dx;\ \pi \int_0^2 (4 - y^2)\,dy$ **3.** $2\pi \int_0^2 (1 + x)x\,dx$ **4.** $2\pi \int_0^2 (1 + y)(2 - y)\,dy$

5.4 Length of a Plane Curve

How long is the spiral curve shown in Figure 1? If it were a piece of string, most of us would stretch it taut and measure it with a ruler. But if it is the graph of an equation, this is a little hard to do.

A little reflection suggests a prior question. What is a plane curve? We have used the term *curve* informally until now, often in reference to the graph of a function. Now it is time to be more precise, even for curves that are not graphs of functions. We begin with several examples.

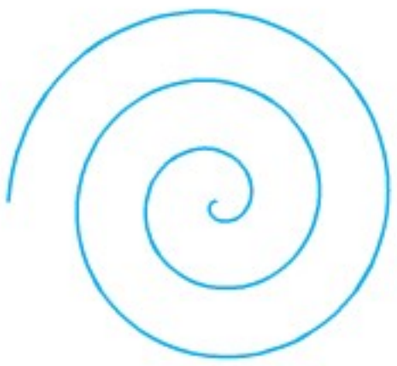


Figure 1

The graph of $y = \sin x$, $0 \le x \le \pi$, is a plane curve (Figure 2). So is the graph of $x = y^2$, $-2 \le y \le 2$ (Figure 3). In both cases, the curve is the graph of a function, the first of the form $y = f(x)$, the second of the form $x = g(y)$. However, the spiral curve does not fit either pattern. Neither does the circle $x^2 + y^2 = a^2$, though in this case we could think of it as the combined graph of the two functions $y = f(x) = \sqrt{a^2 - x^2}$ and $y = g(x) = -\sqrt{a^2 - x^2}$.

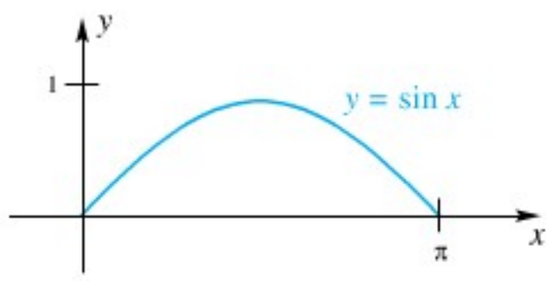


Figure 2

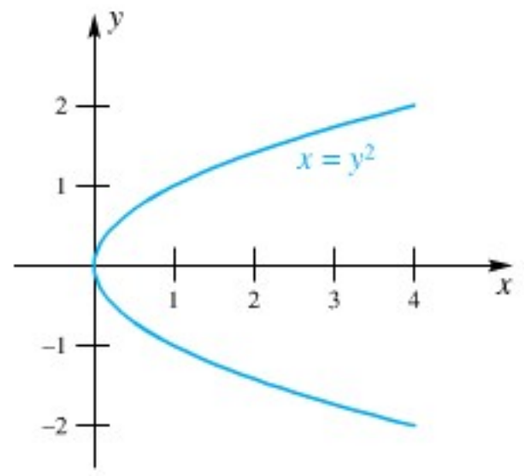


Figure 3

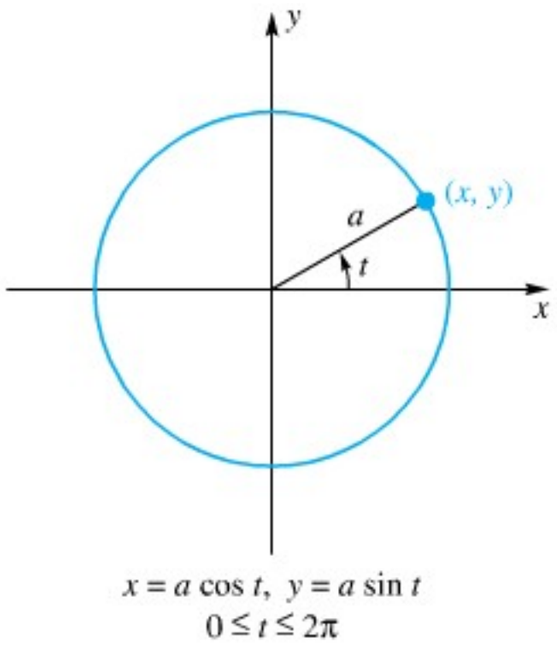


Figure 4

The circle suggests another way of thinking about curves. Recall from trigonometry that

$$x = a\cos t, \quad y = a\sin t, \qquad 0 \le t \le 2\pi$$

describe the circle $x^2 + y^2 = a^2$ (Figure 4). Think of t as time and x and y as giving the position of a particle at time t. The variable t is called a **parameter.** Both x and y are expressed in terms of this parameter. We say that $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$, are **parametric equations** describing the circle.

If we were to graph the parametric equations $x = t\cos t$, $y = t\sin t$, $0 \le t \le 5\pi$, we would get a curve something like the spiral with which we started. And we can even think of the sine curve (Figure 2) and the parabola (Figure 3) in parametric form. We write

$$x = t, \qquad y = \sin t, \qquad 0 \le t \le \pi$$

$x = 2t + 1, y = t^2 - 1$
$0 \le t \le 3$

t	x	y
0	1	−1
1	3	0
2	5	3
3	7	8

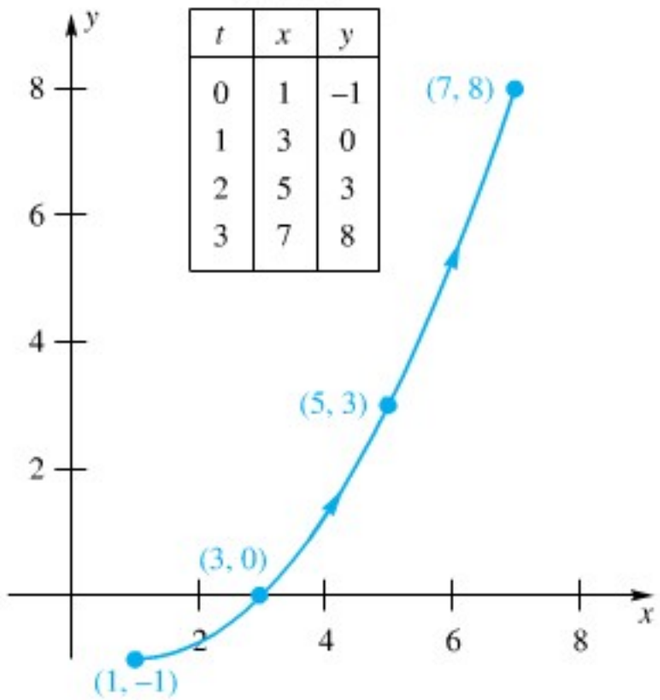


Figure 5

and

$$x = t^2, \qquad y = t, \qquad -2 \le t \le 2$$

Thus, for us, a **plane curve** is determined by a pair of parametric equations $x = f(t), y = g(t), a \le t \le b$, where we assume that f and g are continuous on the given interval. As t increases from a to b, the point (x, y) traces out a curve in the plane. Here is another example.

EXAMPLE 1 Sketch the curve determined by the parametric equations $x = 2t + 1, y = t^2 - 1, 0 \le t \le 3$.

SOLUTION We make a three-column table of values, then plot the ordered pairs (x, y), and finally connect these points in the order of increasing t, as shown in Figure 5. A graphing calculator or a CAS can be used to produce such a graph. Such software usually produces a graph by creating a table, just as we did, and connecting the points. ■

Actually, the definition we have given is too broad for the purposes we have in mind, so we immediately restrict it to what is called a *smooth curve*. The adjective *smooth* is chosen to indicate that as an object moves along the curve so that its position at time t is (x, y) it suffers no sudden changes of direction (continuity of f' and g' ensures this) and does not stop or double back ($f'(t)$ and $g'(t)$ not simultaneously zero ensures this).

Definition

A plane curve is **smooth** if it is determined by a pair of parametric equations $x = f(t), y = g(t), a \le t \le b$, where f' and g' exist and are continuous on $[a, b]$, and $f'(t)$ and $g'(t)$ are not simultaneously zero on (a, b).

The way a curve is parametrized, that is, the way the functions $f(t)$ and $g(t)$, and the domain for t are chosen, determines a *positive direction*. For example, when $t = 0$ in Example 1 (Figure 5), the curve is at the point $(1, -1)$, and when $t = 1$, the curve is at $(3, 0)$. As t increases from $t = 0$ to $t = 3$, the curve traces a path from $(1, -1)$ to $(7, 8)$. This direction, which is often indicated by an arrow on the curve as shown in Figure 5, is called the **orientation** of the curve. The orientation of a curve is irrelevant as far as determining its length goes, but in problems that we will encounter later in this book the orientation does matter.

$x = t - \sin t, y = 1 - \cos t$
$0 \le t \le 4\pi$

t	$x(t)$	$y(t)$
0	0.00	0
$\pi/2$	0.57	1
π	3.14	2
$3\pi/2$	5.71	1
2π	6.28	0
$5\pi/2$	6.85	1
3π	9.42	2
$7\pi/2$	10.00	1
4π	12.57	0

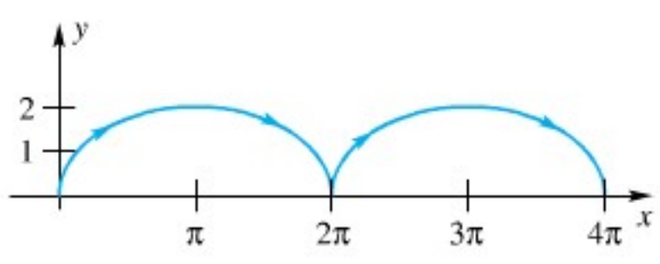


Figure 6

EXAMPLE 2 Sketch the curve determined by the parametric equations $x = t - \sin t, y = 1 - \cos t, 0 \le t \le 4\pi$. Indicate the orientation. Is this curve smooth?

SOLUTION The table, which shows the values of x and y for several values of t from 0 to 4π, leads to the graph in Figure 6. This curve is not smooth even though x and y are both differentiable functions of t. The problem is that $dx/dt = 1 - \cos t$ and $dy/dt = \sin t$ are simultaneously 0 when $t = 2\pi$. The object slows down to a stop at time $t = 2\pi$, then starts up in a new direction. ■

The curve described in Example 2 is called the **cycloid**. It describes the path of a fixed point on the rim of a wheel of radius 1 as the wheel rolls along the x-axis. (See Problem 18 for a derivation of this result.)

Arc Length Finally, we are ready for the main question. What is meant by the length of the smooth curve given parametrically by $x = f(t)$, $y = g(t)$, $a \le t \le b$?

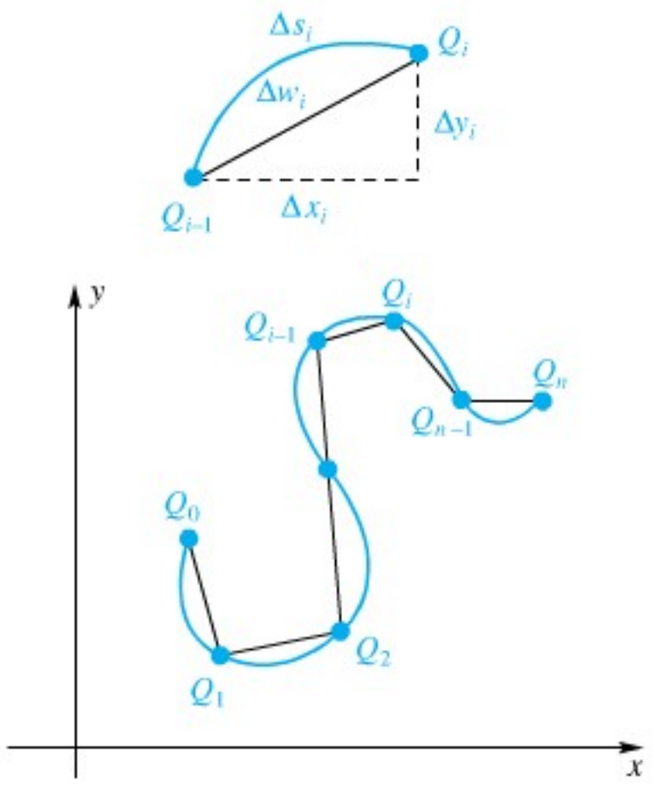


Figure 7

Partition the interval $[a, b]$ into n subintervals by means of points t_i:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

This cuts the curve into n pieces with corresponding end points $Q_0, Q_1, Q_2, \ldots, Q_{n-1}, Q_n$, as shown in Figure 7.

Our idea is to approximate the curve by the indicated polygonal line segments, calculate their total length, and then take the limit as the norm of the partition approaches zero. In particular, we approximate the *length* Δs_i of the ith segment (see Figure 7) by

$$\begin{aligned}\Delta s_i \approx \Delta w_i &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2}\end{aligned}$$

From the Mean Value Theorem for Derivatives (Theorem 3.6A), we know that there are points $\bar{t}_i$ and $\hat{t}_i$ in (t_{i-1}, t_i) such that

$$\begin{aligned}f(t_i) - f(t_{i-1}) &= f'(\bar{t}_i)\,\Delta t_i \\ g(t_i) - g(t_{i-1}) &= g'(\hat{t}_i)\,\Delta t_i\end{aligned}$$

where $\Delta t_i = t_i - t_{i-1}$. Thus,

$$\begin{aligned}\Delta w_i &= \sqrt{[f'(\bar{t}_i)\,\Delta t_i]^2 + [g'(\hat{t}_i)\,\Delta t_i]^2} \\ &= \sqrt{[f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2}\,\Delta t_i\end{aligned}$$

and the total length of the polygonal line segments is

$$\sum_{i=1}^{n} \Delta w_i = \sum_{i=1}^{n} \sqrt{[f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2}\,\Delta t_i$$

The latter expression is almost a Riemann sum, the only difficulty being that $\bar{t}_i$ and $\hat{t}_i$ are not likely to be the same point. However, it is shown in advanced calculus books that in the limit (as the norm of the partition goes to 0), this makes no difference. Thus, we may define the **arc length** L of the curve to be the limit of the expression above as the norm of the partition approaches zero; that is,

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Two special cases are of great interest. If the curve is given by $y = f(x)$, $a \le x \le b$, we treat x as the parameter and the boxed result takes the form

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Similarly, if the curve is given by $x = g(y)$, $c \le y \le d$, we treat y as the parameter, obtaining

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

These formulas yield the familiar results for circles and line segments, as the following two examples illustrate.

EXAMPLE 3 Find the circumference of the circle $x^2 + y^2 = a^2$.

SOLUTION We write the equation of the circle in parametric form: $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$. Then $dx/dt = -a\sin t$, $dy/dt = a\cos t$, and, by the first of our formulas,

$$L = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = \int_0^{2\pi} a\, dt = \Big[at\Big]_0^{2\pi} = 2\pi a$$ ■

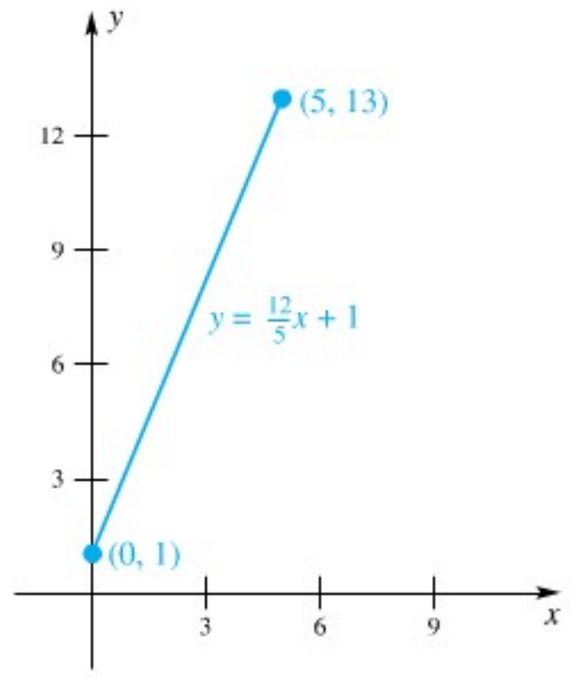


Figure 8

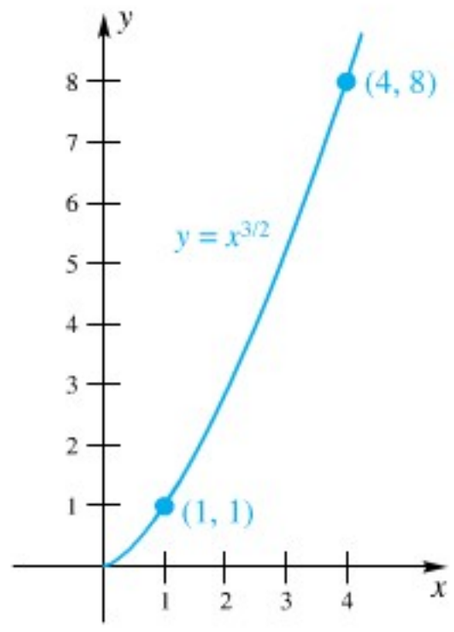


Figure 9

EXAMPLE 4 Find the length of the line segment from $A(0, 1)$ to $B(5, 13)$.

SOLUTION The given line segment is shown in Figure 8. Note that the equation of the corresponding line is $y = \frac{12}{5}x + 1$, so $dy/dx = \frac{12}{5}$; and so, by the second of the three length formulas,

$$L = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2}\, dx = \int_0^5 \sqrt{\frac{5^2 + 12^2}{5^2}}\, dx = \frac{13}{5}\int_0^5 1\, dx$$
$$= \left[\frac{13}{5}x\right]_0^5 = 13$$

This agrees with the result obtained by use of the distance formula. ■

EXAMPLE 5 Find the length of the arc of the curve $y = x^{3/2}$ from the point $(1, 1)$ to the point $(4, 8)$ (see Figure 9).

≈ **SOLUTION** We begin by estimating this length by finding the length of the segment from $(1, 1)$ to $(4, 8)$: $\sqrt{(4-1)^2 + (8-1)^2} = \sqrt{58} \approx 7.6$. The actual length should be slightly larger.

For the exact calculation, we note that $dy/dx = \frac{3}{2}x^{1/2}$, so

$$L = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Let $u = 1 + \frac{9}{4}x$; then $du = \frac{9}{4}dx$. Hence,

$$\int \sqrt{1 + \frac{9}{4}x}\, dx = \frac{4}{9}\int \sqrt{u}\, du = \frac{4}{9}\frac{2}{3}u^{3/2} + C$$
$$= \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2} + C$$

Therefore,

$$\int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_1^4 = \frac{8}{27}\left(10^{3/2} - \frac{13^{3/2}}{8}\right) \approx 7.63$$ ■

For most arc length problems it is easy to set up the definite integral that gives the length. This is just a matter of substituting the required derivatives in the formula. However, it is often difficult or impossible to evaluate these integrals using the Second Fundamental Theorem of Calculus because of the difficulty of finding antiderivatives. For many problems we must resort to using a numerical technique such as the Parabolic Rule described in Section 4.6 in order to obtain an approximation to the definite integral.

EXAMPLE 6 Sketch the graph of the curve given parametrically by $x = 2\cos t$, $y = 4\sin t$, $0 \le t \le \pi$. Set up a definite integral that gives the arc length of the curve and approximate this definite integral using the Parabolic Rule with $n = 8$.

$x = 2\cos t,\ y = 4\sin t$
$0 \le t \le \pi$

t	x	y
0	2	0
$\pi/6$	$\sqrt{3}$	2
$\pi/3$	1	$2\sqrt{3}$
$\pi/2$	0	4
$2\pi/3$	-1	$2\sqrt{3}$
$5\pi/6$	$-\sqrt{3}$	2
π	-2	0

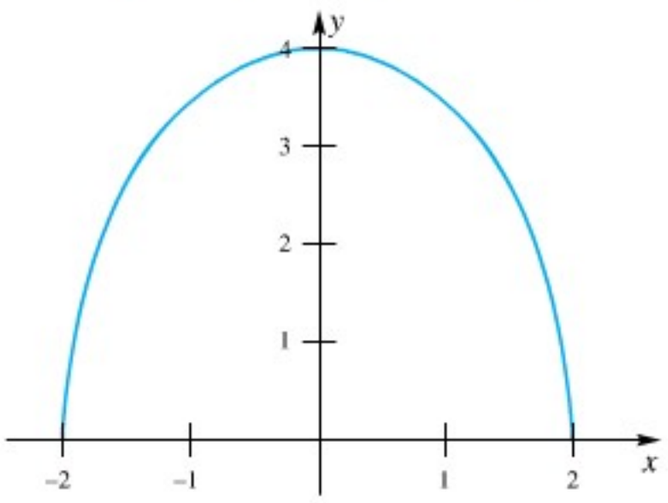


Figure 10

SOLUTION The graph (Figure 10) is drawn, as in previous examples, by first making a three-column table of values. The definite integral that gives the arc length is

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{(-2\sin t)^2 + (4\cos t)^2}\, dt$$

$$= \int_0^{\pi} 2\sqrt{\sin^2 t + 4\cos^2 t}\, dt$$

$$= 2\int_0^{\pi} \sqrt{1 + 3\cos^2 t}\, dt$$

This definite integral cannot be evaluated using the Second Fundamental Theorem of Calculus. Let $f(t) = \sqrt{1 + 3\cos^2 t}$. The approximation using the Parabolic Rule with $n = 8$ is

$$L \approx 2\frac{\pi - 0}{3 \cdot 8}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{4\pi}{8}\right) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{6\pi}{8}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$$

$$\approx 2\frac{\pi}{24}[2.0 + 4 \cdot 1.8870 + 2 \cdot 1.5811 + 4 \cdot 1.1997 + 2 \cdot 1.0 + 4 \cdot 1.1997 + 2 \cdot 1.5811 + 4 \cdot 1.8870 + 2.0)]$$

$$\approx 9.6913$$

■

Differential of Arc Length Let f be continuously differentiable on $[a, b]$. For each x in (a, b), define $s(x)$ by

$$s(x) = \int_a^x \sqrt{1 + [f'(u)]^2}\, du$$

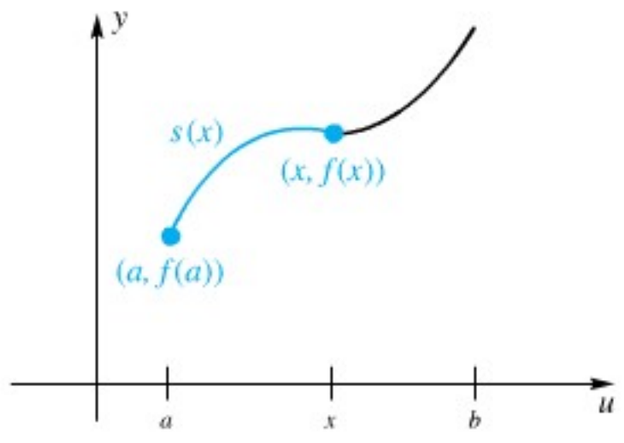


Figure 11

Then $s(x)$ gives the arc length of the curve $y = f(u)$ from the point $(a, f(a))$ to $(x, f(x))$ (see Figure 11). By the First Fundamental Theorem of Calculus (Theorem 4.3A),

$$s'(x) = \frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Thus, ds, the **differential of arc length,** can be written as

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

In fact, depending on how a graph is parametrized, we are led to three formulas for ds:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

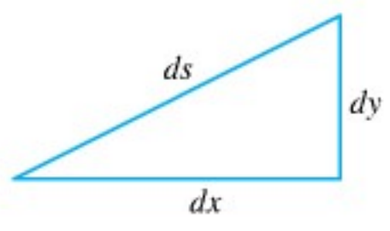


Figure 12

Some people prefer to remember these formulas by writing (see Figure 12)

$$(ds)^2 = (dx)^2 + (dy)^2$$

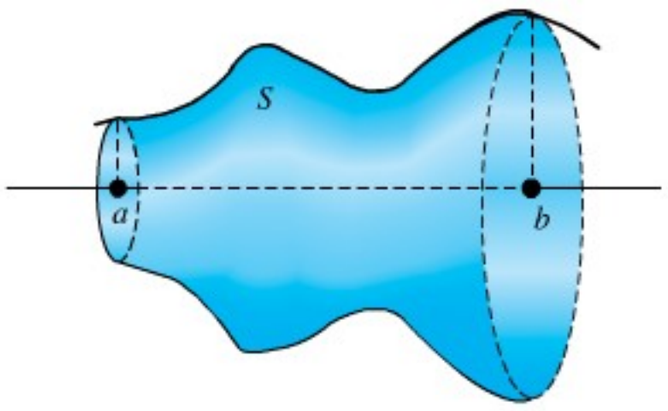


Figure 13

The three forms arise by dividing and multiplying the right-hand side by $(dx)^2$, $(dy)^2$, and $(dt)^2$, respectively. For example,

$$(ds)^2 = \left[\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}\right](dx)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2$$

which gives the first of the three formulas.

Area of a Surface of Revolution If a smooth plane curve is revolved about an axis in its plane, it generates a surface of revolution as illustrated in Figure 13. Our aim is to determine the area of such a surface.

To get started, we introduce the formula for the area of the frustum of a cone. A **frustum** of a cone is that part of the surface of a cone between two planes perpendicular to the axis of the cone (shaded in Figure 14). If a frustum has base radii r_1 and r_2 and slant height ℓ, then its area A is given by

$$A = 2\pi\left(\frac{r_1 + r_2}{2}\right)\ell = 2\pi(\text{average radius}) \cdot (\text{slant height})$$

The derivation of this result depends only on the formula for the area of a circle (see Problem 31).

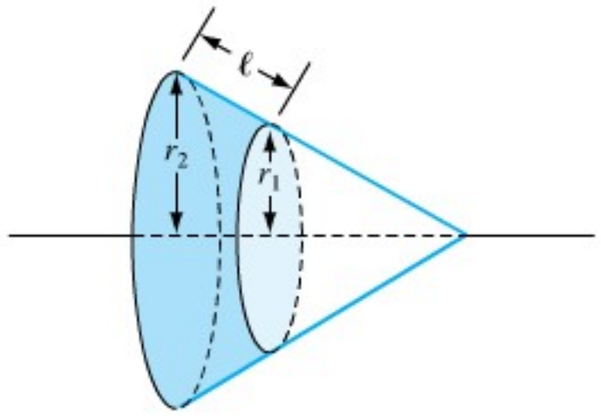


Figure 14

Let $y = f(x)$, $a \le x \le b$, determine a smooth curve in the upper half of the xy-plane, as shown in Figure 15. Partition the interval $[a, b]$ into n pieces by means of points $a = x_0 < x_1 < \cdots < x_n = b$, thereby also dividing the curve into n pieces. Let Δs_i denote the length of the ith piece, and let y_i be the y-coordinate of a point on this piece. When the curve is revolved about the x-axis, it generates a surface, and the typical piece generates a narrow band. The "area" of this band ought to be approximately that of a frustum, that is, approximately $2\pi y_i\,\Delta s_i$. When we add the contributions of all the pieces and take the limit as the norm of the partition approaches zero, we get what we define to be the area of the surface of revolution. All this is indicated in Figure 16. The surface area is thus

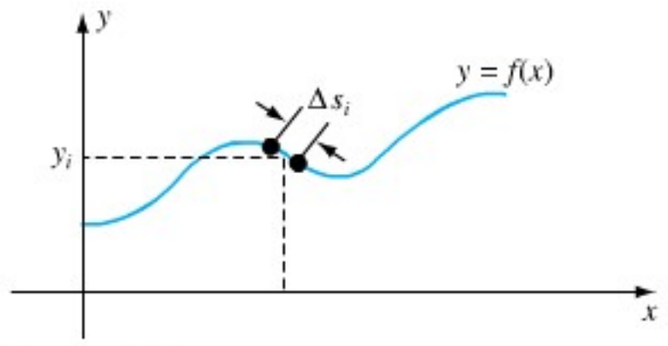


Figure 15

$$\begin{aligned} A &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} 2\pi y_i\,\Delta s_i \\ &= 2\pi \int_a^b y\,ds \\ &= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx \end{aligned}$$

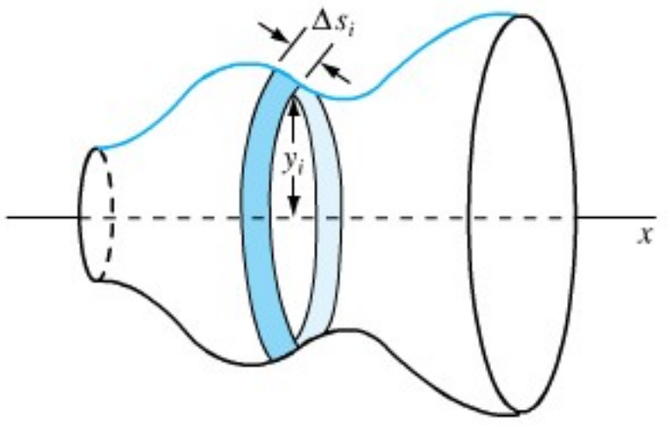


Figure 16

EXAMPLE 7 Find the area of the surface of revolution generated by revolving the curve $y = \sqrt{x}$, $0 \le x \le 4$, about the x-axis (Figure 17).

SOLUTION Here, $f(x) = \sqrt{x}$ and $f'(x) = 1/\left(2\sqrt{x}\right)$. Thus,

$$A = 2\pi \int_0^4 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx = 2\pi \int_0^4 \sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx$$

$$= \pi \int_0^4 \sqrt{4x + 1}\,dx = \left[\pi \cdot \frac{1}{4} \cdot \frac{2}{3}(4x + 1)^{3/2}\right]_0^4$$

$$= \frac{\pi}{6}(17^{3/2} - 1^{3/2}) \approx 36.18$$

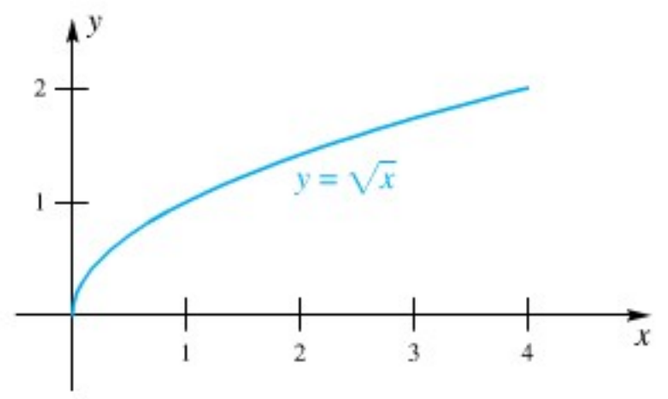


Figure 17

If the curve is given parametrically by $x = f(t)$, $y = g(t)$, $a \le t \le b$, then the surface area formula becomes

$$A = 2\pi \int_a^b y\,ds = 2\pi \int_a^b g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

■

Concepts Review

1. The graph of the parametric equations $x = 4\cos t$, $y = 4\sin t, 0 \le t \le 2\pi$, is a curve called a _____.

2. The curve determined by $y = x^2 + 1, 0 \le x \le 4$, can be put in parametric form using x as the parameter by writing $x =$ _____, $y =$ _____.

3. The formula for the length L of the curve $x = f(t)$, $y = g(t), a \le t \le b$, is $L =$ _____.

4. The proof of the formula for the length of a curve depends strongly on an earlier theorem with the name _____.

Problem Set 5.4

In Problems 1–6, find the length of the indicated curve.

1. $y = 4x^{3/2}$ between $x = 1/3$ and $x = 5$

2. $y = \frac{2}{3}(x^2 + 1)^{3/2}$ between $x = 1$ and $x = 2$

3. $y = (4 - x^{2/3})^{3/2}$ between $x = 1$ and $x = 8$

4. $y = (x^4 + 3)/(6x)$ between $x = 1$ and $x = 3$

5. $x = y^4/16 + 1/(2y^2)$ between $y = -3$ and $y = -2$

Hint: Watch signs; $\sqrt{u^2} = -u$ when $u < 0$.

6. $30xy^3 - y^8 = 15$ between $y = 1$ and $y = 3$

In Problems 7–10, sketch the graph of the given parametric equation and find its length.

7. $x = t^3/3, y = t^2/2; 0 \le t \le 1$

8. $x = 3t^2 + 2, y = 2t^3 - 1/2; 1 \le t \le 4$

9. $x = 4\sin t, y = 4\cos t - 5; 0 \le t \le \pi$

10. $x = \sqrt{5}\sin 2t - 2, y = \sqrt{5}\cos 2t - \sqrt{3}; 0 \le t \le \pi/4$

11. Use an x-integration to find the length of the segment of the line $y = 2x + 3$ between $x = 1$ and $x = 3$. Check by using the distance formula.

12. Use a y-integration to find the length of the segment of the line $2y - 2x + 3 = 0$ between $y = 1$ and $y = 3$. Check by using the distance formula.

In Problems 13–16, set up a definite integral that gives the arc length of the given curve. Approximate the integral using the Parabolic Rule with $n = 8$.

13. $x = t, y = t^2; 0 \le t \le 2$

14. $x = t^2, y = \sqrt{t}; 1 \le t \le 4$

15. $x = \sin t, y = \cos 2t; 0 \le t \le \pi/2$

16. $x = t, y = \tan t; 0 \le t \le \pi/4$

17. Sketch the graph of the four-cusped *hypocycloid* $x = a\sin^3 t, y = a\cos^3 t, 0 \le t \le 2\pi$, and find its length. *Hint:* By symmetry, you can quadruple the length of the first quadrant portion.

18. A point P on the rim of a wheel of radius a is initially at the origin. As the wheel rolls to the right along the x-axis, P traces out a curve called a *cycloid* (see Figure 18). Derive parametric equations for the cycloid as follows. The parameter is θ.

(a) Show that $\overline{OT} = a\theta$.

(b) Convince yourself that $\overline{PQ} = a\sin\theta, \overline{QC} = a\cos\theta$, $0 \le \theta \le \pi/2$.

(c) Show that $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$.

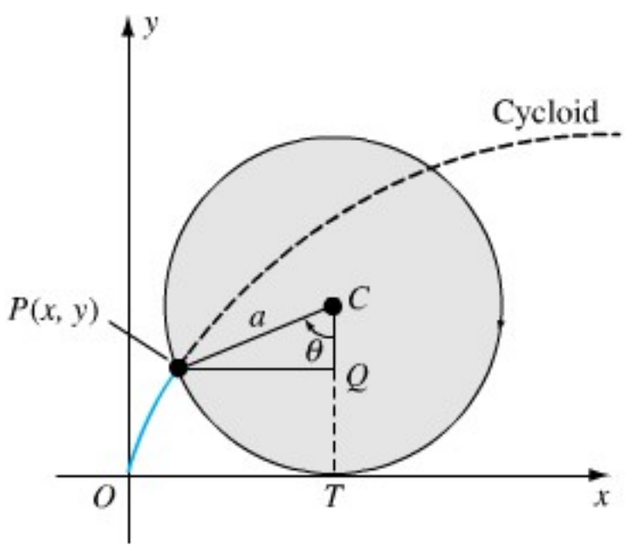


Figure 18

19. Find the length of one arch of the cycloid of Problem 18. *Hint:* First show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4a^2\sin^2\left(\frac{\theta}{2}\right)$$

20. Suppose that the wheel of Problem 18 turns at a constant rate $\omega = d\theta/dt$, where t is time. Then $\theta = \omega t$.

(a) Show that the speed ds/dt of P along the cycloid is

$$\frac{ds}{dt} = 2a\omega\left|\sin\frac{\omega t}{2}\right|$$

(b) When is the speed a maximum and when is it a minimum?

(c) Explain why a bug on a wheel of a car going 60 miles per hour is itself sometimes traveling at 120 miles per hour.

21. Find the length of each curve.

(a) $y = \int_1^x \sqrt{u^3 - 1}\,du, 1 \le x \le 2$

(b) $x = t - \sin t, y = 1 - \cos t, 0 \le t \le 4\pi$

22. Find the length of each curve.

(a) $y = \int_{\pi/6}^x \sqrt{64\sin^2 u\cos^4 u - 1}\,du, \frac{\pi}{6} \le x \le \frac{\pi}{3}$

(b) $x = a\cos t + at\sin t, y = a\sin t - at\cos t, -1 \le t \le 1$

In Problems 23–30, find the area of the surface generated by revolving the given curve about the x-axis.

23. $y = 6x, 0 \le x \le 1$

24. $y = \sqrt{25 - x^2}, -2 \le x \le 3$

25. $y = x^3/3, 1 \le x \le \sqrt{7}$

26. $y = (x^6 + 2)/(8x^2), 1 \le x \le 3$

27. $x = t, y = t^3, 0 \le t \le 1$

CHAPTER 8 Indeterminate Forms and Improper Integrals

8.1 Indeterminate Forms of Type 0/0

8.2 Other Indeterminate Forms

8.3 Improper Integrals: Infinite Limits of Integration

8.4 Improper Integrals: Infinite Integrands

8.1 Indeterminate Forms of Type 0/0

Here are three familiar limit problems:

$$\lim_{x\to 0}\frac{\sin x}{x}, \qquad \lim_{x\to 3}\frac{x^2-9}{x^2-x-6}, \qquad \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

The first was treated at length in Section 1.4, and the third actually defines the derivative $f'(a)$. The three limits have a common feature. In each case, a quotient is involved and, in each case, both numerator and denominator have 0 as their limits. An attempt to apply part 7 of the Main Limit Theorem (Theorem 1.3A), which says that the limit of a quotient is equal to the quotient of the limits, leads to the nonsensical result 0/0. In fact, the theorem does not apply, since it requires that the limit of the denominator be different from 0. We are not saying that these limits do not exist, only that the Main Limit Theorem will not determine them.

You may recall that an intricate geometric argument led us to the conclusion $\lim_{x\to 0}(\sin x)/x = 1$ (Theorem 1.4B). On the other hand, the algebraic technique of factoring yields

$$\lim_{x\to 3}\frac{x^2-9}{x^2-x-6} = \lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)(x+2)} = \lim_{x\to 3}\frac{x+3}{x+2} = \frac{6}{5}$$

Would it not be nice to have a standard procedure for handling all problems for which the limits of the numerator and denominator are both 0? That is too much to hope for. However, there is a simple rule that works beautifully on a wide variety of such problems.

L'Hôpital's Rule In 1696, Guillaume François Antoine de l'Hôpital published the first textbook on differential calculus; it included the following rule, which he had learned from his teacher Johann Bernoulli.

Theorem A **L'Hôpital's Rule for forms of type 0/0**

Suppose that $\lim_{x\to u} f(x) = \lim_{x\to u} g(x) = 0$. If $\lim_{x\to u}[f'(x)/g'(x)]$ exists in either the finite or infinite sense (i.e., if this limit is a finite number or $-\infty$ or $+\infty$), then

$$\lim_{x\to u}\frac{f(x)}{g(x)} = \lim_{x\to u}\frac{f'(x)}{g'(x)}$$

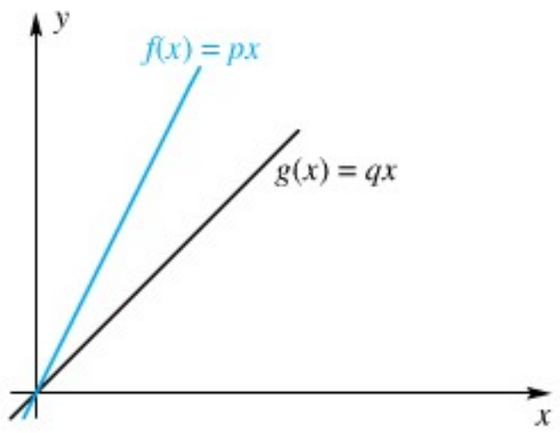
Before attempting to prove this theorem, we illustrate it. Note that l'Hôpital's Rule allows us to replace one limit by another, which may be simpler and, in particular, may not have the 0/0 form.

EXAMPLE 1 Use l'Hôpital's Rule to show that

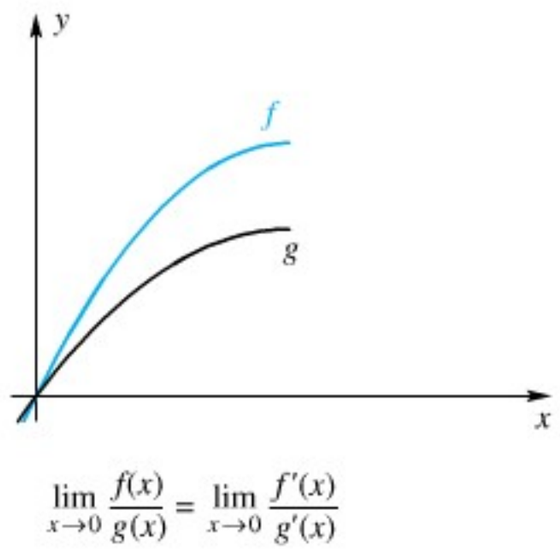
$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

Geometric Interpretation of l'Hôpital's Rule

Study the diagrams below. They should make l'Hôpital's Rule seem quite reasonable. (See Problems 38–42.)



$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{px}{qx} = \frac{p}{q} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$



$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

SOLUTION We worked pretty hard to demonstrate these two facts in Section 1.4. After noting that trying to evaluate both limits by substitution leads to the form 0/0, we can now establish the desired results in two lines (but see Problem 25). By l'Hôpital's Rule,

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{D_x\sin x}{D_x x}=\lim_{x\to 0}\frac{\cos x}{1}=1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{D_x(1-\cos x)}{D_x x}=\lim_{x\to 0}\frac{\sin x}{1}=0$$ ■

EXAMPLE 2 Find $\lim_{x\to 3}\dfrac{x^2-9}{x^2-x-6}$ and $\lim_{x\to 2^+}\dfrac{x^2+3x-10}{x^2-4x+4}$.

SOLUTION Both limits have the 0/0 form, so, by l'Hôpital's Rule,

$$\lim_{x\to 3}\frac{x^2-9}{x^2-x-6}=\lim_{x\to 3}\frac{2x}{2x-1}=\frac{6}{5}$$

$$\lim_{x\to 2^+}\frac{x^2+3x-10}{x^2-4x+4}=\lim_{x\to 2^+}\frac{2x+3}{2x-4}=\infty$$

The first of these limits was handled at the beginning of this section by factoring and simplifying. Of course, we get the same answer either way. ■

EXAMPLE 3 Find $\lim_{x\to 0}\dfrac{\tan 2x}{\ln(1+x)}$.

SOLUTION Both numerator and denominator have limit 0. Hence,

$$\lim_{x\to 0}\frac{\tan 2x}{\ln(1+x)}=\lim_{x\to 0}\frac{2\sec^2 2x}{1/(1+x)}=\frac{2}{1}=2$$ ■

Sometimes $\lim f'(x)/g'(x)$ also has the indeterminate form 0/0. Then we may apply l'Hôpital's Rule again, as we now illustrate. Each application of l'Hôpital's Rule is flagged with the symbol Ⓛ

EXAMPLE 4 Find $\lim_{x\to 0}\dfrac{\sin x-x}{x^3}$.

SOLUTION By l'Hôpital's Rule applied three times in succession,

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}\overset{\text{Ⓛ}}{=}\lim_{x\to 0}\frac{\cos x-1}{3x^2}$$

$$\overset{\text{Ⓛ}}{=}\lim_{x\to 0}\frac{-\sin x}{6x}$$

$$\overset{\text{Ⓛ}}{=}\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$$ ■

Just because we have an elegant rule does not mean that we should use it indiscriminately. In particular, we must always make sure that it applies; that is, we must make sure that the limit has the indeterminate form 0/0. Otherwise, we will be led into all kinds of errors, as we now illustrate.

EXAMPLE 5 Find $\lim_{x\to 0} \dfrac{1-\cos x}{x^2+3x}$.

SOLUTION We might be tempted to write

$$\lim_{x\to 0}\frac{1-\cos x}{x^2+3x} \overset{\text{L}}{=} \lim_{x\to 0}\frac{\sin x}{2x+3} \overset{\text{L}}{=} \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2} \quad \text{WRONG}$$

The first application of l'Hôpital's Rule was correct; the second was not, since at that stage, the limit did not have the 0/0 form. Here is what we should have done.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2+3x} \overset{\text{L}}{=} \lim_{x\to 0}\frac{\sin x}{2x+3} = 0 \quad \text{RIGHT}$$

We stop differentiating as soon as either the numerator or denominator has a nonzero limit. ■

Even if the conditions of l'Hôpital's Rule hold, an application of l'Hôpital's Rule may not help us; witness the following example.

EXAMPLE 6 Find $\lim_{x\to\infty} \dfrac{e^{-x}}{x^{-1}}$.

SOLUTION Since the numerator and denominator both tend to 0, the limit is indeterminate of the form 0/0. Thus, the conditions of Theorem A hold. We may apply l'Hôpital's Rule indefinitely.

$$\lim_{x\to\infty}\frac{e^{-x}}{x^{-1}} \overset{\text{L}}{=} \lim_{x\to\infty}\frac{e^{-x}}{x^{-2}} \overset{\text{L}}{=} \lim_{x\to\infty}\frac{e^{-x}}{2x^{-3}} = \cdots$$

Clearly, we are only complicating the problem. A better approach is to do a bit of algebra first.

$$\lim_{x\to\infty}\frac{e^{-x}}{x^{-1}} = \lim_{x\to\infty}\frac{x}{e^{x}}$$

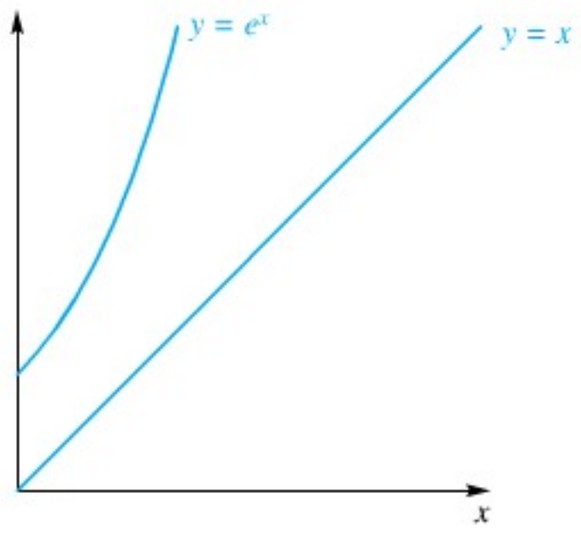


Figure 1

Written this way, the limit is indeterminate of the form ∞/∞, the subject of the next section. However, you should be able to guess that the limit is 0 by considering how much faster e^x grows than x (see Figure 1). A rigorous demonstration will come later (Example 1 of Section 8.2). ■

Cauchy's Mean Value Theorem The proof of l'Hôpital's Rule depends on an extension of the Mean Value Theorem for Derivatives due to Augustin Louis Cauchy (1789–1857).

Theorem B Cauchy's Mean Value Theorem

Let the functions f and g be differentiable on (a, b) and continuous on $[a, b]$. If $g'(x) \neq 0$ for all x in (a, b), then there exists a number c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Note that this theorem reduces to the ordinary Mean Value Theorem for Derivatives (Theorem 3.6A) when $g(x) = x$.

Proof It is tempting to apply the ordinary Mean Value Theorem to both numerator and denominator of the left side of the conclusion. If we do this, we obtain

$$f(b) - f(a) = f'(c_1)(b - a) \tag{1}$$

and

$$g(b) - g(a) = g'(c_2)(b - a) \tag{2}$$

for appropriate choices of c_1 and c_2. If only c_1 and c_2 were equal, we could divide the first equality by the second and be done; but there is no reason to hope for such a coincidence. However, this attempt is not a complete failure since (2) yields the valuable information that $g(b) - g(a) \neq 0$, a fact we will need later (this follows from the hypothesis that $g'(x) \neq 0$ for all x in (a, b)).

Recall that the proof of the Mean Value Theorem for Derivatives (Theorem 3.6A) rested on the introduction of an auxiliary function s. If we try to mimic that proof, we are led to the following choice for $s(x)$. Let

$$s(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

No division by zero is involved since we earlier established that $g(b) - g(a) \neq 0$. Note further that $s(a) = 0 = s(b)$. Also, s is continuous on $[a, b]$ and differentiable on (a, b), this following from the corresponding facts for f and g. Thus, by the Mean Value Theorem for Derivatives, there is a number c in (a, b) such that

$$s'(c) = \frac{s(b) - s(a)}{b - a} = \frac{0 - 0}{b - a} = 0$$

But

$$s'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) = 0$$

so

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

which is what we wished to prove. ■

Proof of L'Hôpital's Rule

Proof Refer back to Theorem A, which actually states several theorems at once. We will prove only the case where L is finite and the limit is the one-sided limit $\lim_{x \to a^+}$.

The hypotheses for Theorem A imply more than they say explicitly. In particular, the existence of $\lim_{x \to a^+}[f'(x)/g'(x)]$ implies that both $f'(x)$ and $g'(x)$ exist in at least a small interval $(a, b]$ and that $g'(x) \neq 0$ there. At a, we do not even know that f and g are defined, but we do know that $\lim_{x \to a^+} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$. Thus, we may define (or redefine if necessary) both $f(a)$ and $g(a)$ to be zero, thereby making both f and g (right) continuous at a. All this is to say that f and g satisfy the hypotheses of Cauchy's Mean Value Theorem on $[a, b]$. Consequently, there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

or, since $f(a) = 0 = g(a)$,

$$\frac{f(b)}{g(b)} = \frac{f'(c)}{g'(c)}$$

When we let $b \to a^+$, thereby forcing $c \to a^+$, we obtain

$$\lim_{b\to a^+} \frac{f(b)}{g(b)} = \lim_{c\to a^+} \frac{f'(c)}{g'(c)}$$

which is equivalent to what we wanted to prove.

A very similar proof works for the case of left-hand limits and thus for two-sided limits. The proofs for limits at infinity and infinite limits are harder, and we omit them. ■

Concepts Review

1. L'Hôpital's Rule is useful in finding $\lim_{x\to a}[f(x)/g(x)]$, where both _____ and _____ are zero.

2. L'Hôpital's Rule says that under appropriate conditions $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a}$ _____.

3. From l'Hôpital's Rule, we can conclude that $\lim_{x\to 0}(\tan x)/x = \lim_{x\to 0}$ _____ = _____, but l'Hôpital's Rule gives us no information about $\lim_{x\to 0}(\cos x)/x$ because _____.

4. The proof of l'Hôpital's Rule depends on _____ Theorem.

Problem Set 8.1

In Problems 1–24, find the indicated limit. Make sure that you have an indeterminate form before you apply l'Hôpital's Rule.

1. $\lim_{x\to 0} \dfrac{2x - \sin x}{x}$

2. $\lim_{x\to \pi/2} \dfrac{\cos x}{\frac{1}{2}\pi - x}$

3. $\lim_{x\to 0} \dfrac{x - \sin 2x}{\tan x}$

4. $\lim_{x\to 0} \dfrac{\tan^{-1} 3x}{\sin^{-1} x}$

5. $\lim_{x\to -2} \dfrac{x^2 + 6x + 8}{x^2 - 3x - 10}$

6. $\lim_{x\to 0} \dfrac{x^3 - 3x^2 + x}{x^3 - 2x}$

7. $\lim_{x\to 1^-} \dfrac{x^2 - 2x + 2}{x^2 - 1}$

8. $\lim_{x\to 1} \dfrac{\ln x^2}{x^2 - 1}$

9. $\lim_{x\to \pi/2} \dfrac{\ln(\sin x)^3}{\frac{1}{2}\pi - x}$

10. $\lim_{x\to 0} \dfrac{e^x - e^{-x}}{2 \sin x}$

11. $\lim_{t\to 1} \dfrac{\sqrt{t} - t^2}{\ln t}$

12. $\lim_{x\to 0^+} \dfrac{7^{\sqrt{x}} - 1}{2^{\sqrt{x}} - 1}$

13. $\lim_{x\to 0} \dfrac{\ln \cos 2x}{7x^2}$

14. $\lim_{x\to 0^-} \dfrac{3 \sin x}{\sqrt{-x}}$

15. $\lim_{x\to 0} \dfrac{\tan x - x}{\sin 2x - 2x}$

16. $\lim_{x\to 0} \dfrac{\sin x - \tan x}{x^2 \sin x}$

17. $\lim_{x\to 0^+} \dfrac{x^2}{\sin x - x}$

18. $\lim_{x\to 0} \dfrac{e^x - \ln(1 + x) - 1}{x^2}$

19. $\lim_{x\to 0} \dfrac{\tan^{-1} x - x}{8x^3}$

20. $\lim_{x\to 0} \dfrac{\cosh x - 1}{x^2}$

21. $\lim_{x\to 0^+} \dfrac{1 - \cos x - x \sin x}{2 - 2\cos x - \sin^2 x}$

22. $\lim_{x\to 0^-} \dfrac{\sin x + \tan x}{e^x + e^{-x} - 2}$

23. $\lim_{x\to 0} \dfrac{\int_0^x \sqrt{1 + \sin t}\, dt}{x}$

24. $\lim_{x\to 0^+} \dfrac{\int_0^x \sqrt{t} \cos t\, dt}{x^2}$

25. In Section 1.4, we worked very hard to prove that $\lim_{x\to 0}(\sin x)/x = 1$; l'Hôpital's Rule allows us to show this in one line. However, even if we had l'Hôpital's Rule, say at the end of Section 1.3, it would not have helped us. Explain why. (We really did need to establish $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ the way we did in Section 1.4.)

26. Find $\lim_{x\to 0} \dfrac{x^2 \sin(1/x)}{\tan x}$.

Hint: Begin by deciding why l'Hôpital's Rule is not applicable. Then find the limit by other means.

27. For Figure 2, compute the following limits.

(a) $\lim_{t\to 0^+} \dfrac{\text{area of triangle } ABC}{\text{area of curved region } ABC}$

(b) $\lim_{t\to 0^+} \dfrac{\text{area of curved region } BCD}{\text{area of curved region } ABC}$

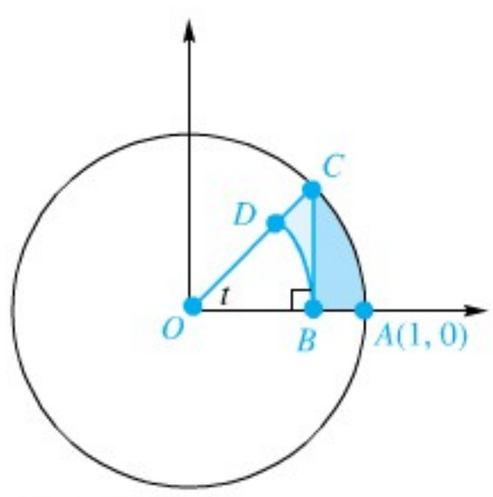


Figure 2

28. In Figure 3, $CD = DE = DF = t$. Find each limit.

(a) $\lim_{t \to 0^+} y$ (b) $\lim_{t \to 0^+} x$

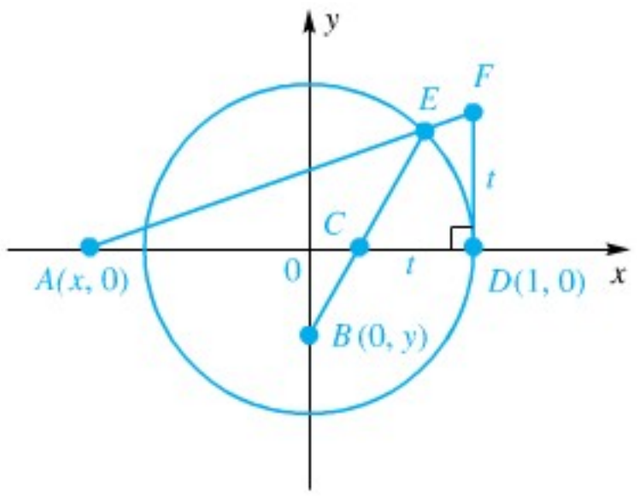


Figure 3

29. Let

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & \text{if } x \neq 0 \\ c, & \text{if } x = 0 \end{cases}$$

What value of c makes $f(x)$ continuous at $x = 0$?

30. Let

$$f(x) = \begin{cases} \dfrac{\ln x}{x - 1}, & \text{if } x \neq 1 \\ c, & \text{if } x = 1 \end{cases}$$

What value of c makes $f(x)$ continuous at $x = 1$?

31. Using the concepts of Section 5.4, you can show that the surface area of the prolate spheroid gotten by rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$ about the x-axis is

$$A = 2\pi b^2 + 2\pi ab\left[\frac{a}{\sqrt{a^2 - b^2}} \arcsin \frac{\sqrt{a^2 - b^2}}{a}\right]$$

What should A approach as $a \to b^+$? Use l'Hôpital's Rule to show that this does happen.

32. Determine constants a, b, and c so that

$$\lim_{x \to 1} \frac{ax^4 + bx^3 + 1}{(x - 1)\sin \pi x} = c$$

33. L'Hôpital's Rule in its 1696 form said this: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x)/g(x) = f'(a)/g'(a)$, provided that $f'(a)$ and $g'(a)$ both exist and $g'(a) \neq 0$. Prove this result without recourse to Cauchy's Mean Value Theorem.

CAS *Use a CAS to evaluate the limits in Problems 34–37.*

34. $\lim_{x \to 0} \dfrac{\cos x - 1 + x^2/2}{x^4}$

35. $\lim_{x \to 0} \dfrac{e^x - 1 - x - x^2/2 - x^3/6}{x^4}$

36. $\lim_{x \to 0} \dfrac{1 - \cos(x^2)}{x^3 \sin x}$

37. $\lim_{x \to 0} \dfrac{\tan x - x}{\arcsin x - x}$

GC *For Problems 38–41, plot the numerator $f(x)$ and the denominator $g(x)$ in the same graph window for each of these domains: $-1 \le x \le 1$, $-0.1 \le x \le 0.1$, and $-0.01 \le x \le 0.01$. From the plot, estimate the values of $f'(x)$ and $g'(x)$ and use these to approximate the given limit.*

38. $\lim_{x \to 0} \dfrac{3x - \sin x}{x}$

39. $\lim_{x \to 0} \dfrac{\sin x/2}{x}$

40. $\lim_{x \to 0} \dfrac{x}{e^{2x} - 1}$

41. $\lim_{x \to 0} \dfrac{e^x - 1}{e^{-x} - 1}$

EXPL **42.** Use the concept of the **linear approximation** to a function (Section 2.9) to explain the geometric interpretation of l'Hôpital's Rule in the marginal box next to Theorem A.

Answers to Concepts Review: **1.** $\lim_{x \to a} f(x)$; $\lim_{x \to a} g(x)$ **2.** $f'(x)/g'(x)$ **3.** $\sec^2 x$; 1; $\lim_{x \to 0} \cos x \neq 0$ **4.** Cauchy's Mean Value

8.2 Other Indeterminate Forms

In the solution to Example 6 of the previous section, we faced the following limit problem.

$$\lim_{x \to \infty} \frac{x}{e^x}$$

This is typical of a class of problems of the form $\lim_{x \to \infty} f(x)/g(x)$, where both numerator and denominator are growing indefinitely large; we call it an indeterminate form of type ∞/∞. It turns out that l'Hôpital's Rule also applies in this situation; that is,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

A rigorous proof is quite difficult, but there is an intuitive way of seeing that the result has to be true. Imagine that $f(t)$ and $g(t)$ represent the positions of two cars on the t-axis at time t (Figure 1). These two cars, the f-car and the g-car, are on endless journeys with respective velocities $f'(t)$ and $g'(t)$. Now, if

$$\lim_{t \to \infty} \frac{f'(t)}{g'(t)} = L$$

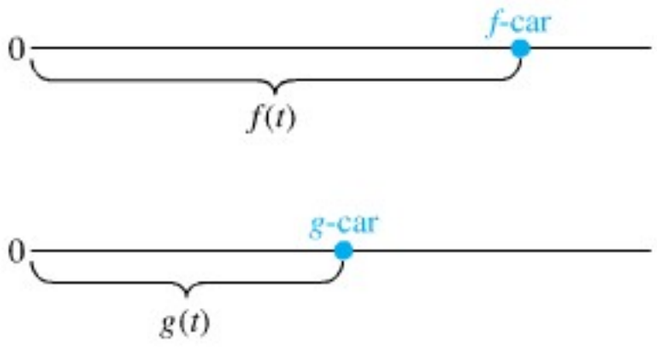


Figure 1

then ultimately the f-car travels about L times as fast as the g-car. It is therefore reasonable to say that, in the long run, it will travel about L times as far; that is,

$$\lim_{t\to\infty}\frac{f(t)}{g(t)} = L$$

We do not call this a proof, but it does lend plausibility to a result that we now state formally.

Theorem A **L'Hôpital's Rule for Forms of Type ∞/∞**

Suppose that $\lim_{x\to u}|f(x)| = \lim_{x\to u}|g(x)| = \infty$. If $\lim_{x\to u}[f'(x)/g'(x)]$ exists in either the finite or infinite sense, then

$$\lim_{x\to u}\frac{f(x)}{g(x)} = \lim_{x\to u}\frac{f'(x)}{g'(x)}$$

Here u may stand for any of the symbols a, a^-, a^+, $-\infty$, or ∞.

The Indeterminate Form ∞/∞ We use Theorem A to finish Example 6 of the previous section.

EXAMPLE 1 Find $\lim_{x\to\infty}\frac{x}{e^x}$.

SOLUTION Both x and e^x tend to ∞ as $x\to\infty$. Hence, by l'Hôpital's Rule,

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{D_x x}{D_x e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$$ ■

Here is a general result of the same type.

EXAMPLE 2 Show that, if a is any positive real number, $\lim_{x\to\infty}\frac{x^a}{e^x} = 0$.

SOLUTION Suppose as a special case that $a = 2.5$. Then three applications of l'Hôpital's Rule give

$$\lim_{x\to\infty}\frac{x^{2.5}}{e^x} \overset{\text{L}}{=} \lim_{x\to\infty}\frac{2.5x^{1.5}}{e^x} \overset{\text{L}}{=} \lim_{x\to\infty}\frac{(2.5)(1.5)x^{0.5}}{e^x} \overset{\text{L}}{=} \lim_{x\to\infty}\frac{(2.5)(1.5)(0.5)}{x^{0.5}e^x} = 0$$

A similar argument works for any $a > 0$. Let m denote the greatest integer less than a. Then $m + 1$ applications of l'Hôpital's Rule give

$$\lim_{x\to\infty}\frac{x^a}{e^x} \overset{\text{L}}{=} \lim_{x\to\infty}\frac{ax^{a-1}}{e^x} \overset{\text{L}}{=} \lim_{x\to\infty}\frac{a(a-1)x^{a-2}}{e^x} \overset{\text{L}}{=} \cdots \overset{\text{L}}{=} \lim_{x\to\infty}\frac{a(a-1)\cdots(a-m)}{x^{m+1-a}e^x} = 0$$ ■

EXAMPLE 3 Show that, if a is any positive real number, $\lim_{x\to\infty}\frac{\ln x}{x^a} = 0$.

See How They Grow

In computer science, one pays careful attention to the amount of time needed to perform a task. For example, to sort x items using the "bubble sort" algorithm takes time proportional to x^2, whereas the "quick sort" algorithm does the same task in time proportional to $x \ln x$, a major improvement. Here is a chart illustrating how some common functions grow as x increases from 10 to 100 to 1000.

$\ln x$	2.3	4.6	6.9
$\sqrt{x}$	3.2	10	31.6
x	10	100	1000
$x \ln x$	23	461	6908
x^2	100	10000	10^6
e^x	2.2×10^4	2.7×10^{43}	10^{434}

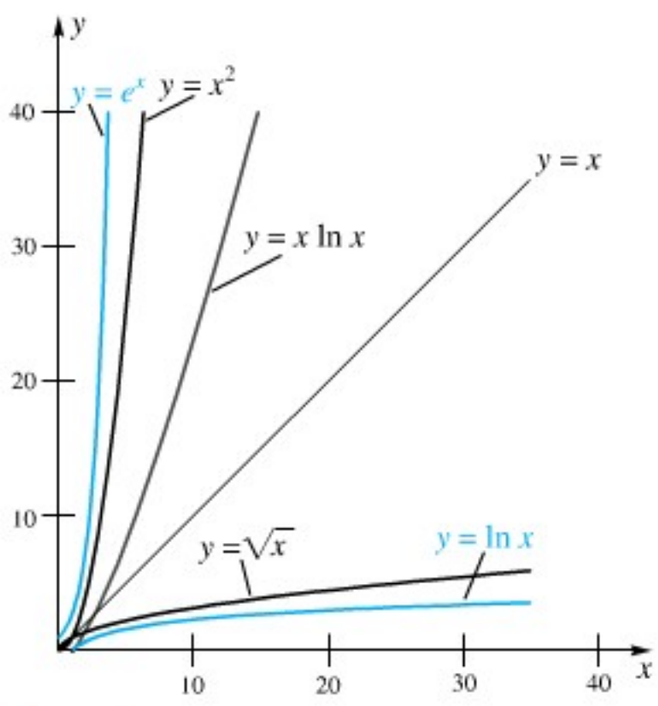


Figure 2

SOLUTION Both $\ln x$ and x^a tend to ∞ as $x \to \infty$. Hence, by one application of l'Hôpital's Rule,

$$\lim_{x \to \infty} \frac{\ln x}{x^a} \overset{\text{L}}{=} \lim_{x \to \infty} \frac{1/x}{ax^{a-1}} = \lim_{x \to \infty} \frac{1}{ax^a} = 0$$

■

≈ Examples 2 and 3 say something that is worth remembering: *for sufficiently large x, e^x grows faster as x increases than any constant power of x, whereas $\ln x$ grows more slowly than any constant power of x.* For example, when x is sufficiently large, e^x grows faster than x^{100} and $\ln x$ grows more slowly than $\sqrt[100]{x}$. The chart in the margin and Figure 2 offer additional illustration.

EXAMPLE 4 Find $\lim_{x \to 0^+} \dfrac{\ln x}{\cot x}$.

SOLUTION As $x \to 0^+$, $\ln x \to -\infty$ and $\cot x \to \infty$, so l'Hôpital's Rule applies.

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} \overset{\text{L}}{=} \lim_{x \to 0^+} \left[\frac{1/x}{-\csc^2 x}\right]$$

This is still indeterminate as it stands, but rather than apply l'Hôpital's Rule again (which only makes things worse), we rewrite the expression in brackets as

$$\frac{1/x}{-\csc^2 x} = -\frac{\sin^2 x}{x} = -\sin x \frac{\sin x}{x}$$

Thus,

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = \lim_{x \to 0^+} \left[-\sin x \frac{\sin x}{x}\right] = 0 \cdot 1 = 0$$

■

The Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$ Suppose that $A(x) \to 0$, but $B(x) \to \infty$. What is going to happen to the product $A(x)B(x)$? Two competing forces are at work, tending to pull the product in opposite directions. Which will win this battle, A or B or neither? It depends on whether one is stronger (i.e., doing its job at a faster rate) or whether they are evenly matched. L'Hôpital's Rule will help us to decide, but only after we have transformed the problem to a $0/0$ or ∞/∞ form.

EXAMPLE 5 Find $\lim_{x \to \pi/2} (\tan x \cdot \ln \sin x)$.

SOLUTION Since $\lim_{x \to \pi/2} \ln \sin x = 0$ and $\lim_{x \to \pi/2} |\tan x| = \infty$, this is a $0 \cdot \infty$ indeterminate form. We can rewrite it as a $0/0$ form by the simple device of changing $\tan x$ to $1/\cot x$. Thus,

$$\lim_{x \to \pi/2} (\tan x \cdot \ln \sin x) = \lim_{x \to \pi/2} \frac{\ln \sin x}{\cot x}$$

$$\overset{\text{L}}{=} \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc^2 x}$$

$$= \lim_{x \to \pi/2} (-\cos x \cdot \sin x) = 0$$

■

EXAMPLE 6 Find $\lim_{x\to 1^+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$.

SOLUTION The first term is growing without bound; so is the second. We say that the limit is an $\infty - \infty$ indeterminate form. L'Hôpital's Rule will determine the result, but only after we rewrite the problem in a form for which the rule applies. In this case, the two fractions must be combined, a procedure that changes the problem to a 0/0 form. Two applications of l'Hôpital's Rule yield

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\frac{x\ln x-x+1}{(x-1)\ln x}\overset{\text{L}}{=}\lim_{x\to 1^+}\frac{x\cdot 1/x+\ln x-1}{(x-1)(1/x)+\ln x}$$

$$=\lim_{x\to 1^+}\frac{x\ln x}{x-1+x\ln x}\overset{\text{L}}{=}\lim_{x\to 1^+}\frac{1+\ln x}{2+\ln x}=\frac{1}{2}$$

■

The Indeterminate Forms 0^0, ∞^0, 1^∞ We turn now to three indeterminate forms of exponential type. Here the trick is to consider not the original expression, but rather its logarithm. Usually, l'Hôpital's Rule will apply to the logarithm.

EXAMPLE 7 Find $\lim_{x\to 0^+}(x+1)^{\cot x}$.

SOLUTION This takes the indeterminate form 1^∞. Let $y=(x+1)^{\cot x}$, so

$$\ln y=\cot x\ln(x+1)=\frac{\ln(x+1)}{\tan x}$$

Using l'Hôpital's Rule for 0/0 forms, we obtain

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(x+1)}{\tan x}\overset{\text{L}}{=}\lim_{x\to 0^+}\frac{\dfrac{1}{x+1}}{\sec^2 x}=1$$

Now $y=e^{\ln y}$, and since the exponential function $f(x)=e^x$ is continuous,

$$\lim_{x\to 0^+}y=\lim_{x\to 0^+}\exp(\ln y)=\exp\left(\lim_{x\to 0^+}\ln y\right)=\exp 1=e$$

■

EXAMPLE 8 Find $\lim_{x\to \pi/2^-}(\tan x)^{\cos x}$.

SOLUTION This has the indeterminate form ∞^0. Let $y=(\tan x)^{\cos x}$, so

$$\ln y=\cos x\cdot\ln\tan x=\frac{\ln\tan x}{\sec x}$$

Then

$$\lim_{x\to \pi/2^-}\ln y=\lim_{x\to \pi/2^-}\frac{\ln\tan x}{\sec x}\overset{\text{L}}{=}\lim_{x\to \pi/2^-}\frac{\dfrac{1}{\tan x}\cdot\sec^2 x}{\sec x\tan x}$$

$$=\lim_{x\to \pi/2^-}\frac{\sec x}{\tan^2 x}=\lim_{x\to \pi/2^-}\frac{\cos x}{\sin^2 x}=0$$

Therefore,

$$\lim_{x \to \pi/2^-} y = e^0 = 1$$ ■

Summary We have classified certain limit problems as indeterminate forms, using the seven symbols $0/0$, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, and 1^∞. Each involves a competition of opposing forces, which means that the result is not obvious. However, with the help of l'Hôpital's Rule, which applies directly only to the $0/0$ and ∞/∞ forms, we can usually determine the limit.

There are many other possibilities symbolized by, for example, $0/\infty$, $\infty/0$, $\infty + \infty$, $\infty \cdot \infty$, 0^∞, and ∞^∞. Why don't we call these indeterminate forms? Because, in each of these cases, the forces work together, not in competition.

EXAMPLE 9 Find $\lim_{x \to 0^+} (\sin x)^{\cot x}$.

SOLUTION We might call this a 0^∞ form, but it is not indeterminate. Note that $\sin x$ is approaching zero, and raising it to the exponent $\cot x$, an increasingly large number, serves only to make it approach zero faster. Thus,

$$\lim_{x \to 0^+} (\sin x)^{\cot x} = 0$$ ■

Concepts Review

1. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$, then l'Hôpital's Rule says that $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a}$ _____.

2. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} f(x)g(x)$ is an indeterminate form. To apply l'Hôpital's Rule, we may rewrite this latter limit as _____.

3. Seven indeterminate forms are discussed in this book. They are symbolized by $0/0$, ∞/∞, $0 \cdot \infty$, and _____.

4. e^x grows faster than any power of x, but _____ grows more slowly than any power of x.

Problem Set 8.2

Find each limit in Problems 1–40. Be sure you have an indeterminate form before applying l'Hôpital's Rule.

1. $\lim_{x \to \infty} \dfrac{\ln x^{10000}}{x}$

2. $\lim_{x \to \infty} \dfrac{(\ln x)^2}{2^x}$

3. $\lim_{x \to \infty} \dfrac{x^{10000}}{e^x}$

4. $\lim_{x \to \infty} \dfrac{3x}{\ln(100x + e^x)}$

5. $\lim_{x \to \pi/2} \dfrac{3 \sec x + 5}{\tan x}$

6. $\lim_{x \to 0^+} \dfrac{\ln \sin^2 x}{3 \ln \tan x}$

7. $\lim_{x \to \infty} \dfrac{\ln(\ln x^{1000})}{\ln x}$

8. $\lim_{x \to (1/2)^-} \dfrac{\ln(4 - 8x)^2}{\tan \pi x}$

9. $\lim_{x \to 0^+} \dfrac{\cot x}{\sqrt{-\ln x}}$

10. $\lim_{x \to 0} \dfrac{2 \csc^2 x}{\cot^2 x}$

11. $\lim_{x \to 0} (x \ln x^{1000})$

12. $\lim_{x \to 0} 3x^2 \csc^2 x$

13. $\lim_{x \to 0} (\csc^2 x - \cot^2 x)$

14. $\lim_{x \to \pi/2} (\tan x - \sec x)$

15. $\lim_{x \to 0^+} (3x)^{x^2}$

16. $\lim_{x \to 0} (\cos x)^{\csc x}$

17. $\lim_{x \to (\pi/2)^-} (5 \cos x)^{\tan x}$

18. $\lim_{x \to 0} \left(\csc^2 x - \dfrac{1}{x^2} \right)^2$

19. $\lim_{x \to 0} (x + e^{x/3})^{3/x}$

20. $\lim_{x \to (\pi/2)^-} (\cos 2x)^{x - \pi/2}$

21. $\lim_{x \to \pi/2} (\sin x)^{\cos x}$

22. $\lim_{x \to \infty} x^x$

23. $\lim_{x \to \infty} x^{1/x}$

24. $\lim_{x \to 0} (\cos x)^{1/x^2}$

25. $\lim_{x \to 0^+} (\tan x)^{2/x}$

26. $\lim_{x \to -\infty} (e^{-x} - x)$

27. $\lim_{x \to 0^+} (\sin x)^x$

28. $\lim_{x \to 0} (\cos x - \sin x)^{1/x}$

29. $\lim_{x \to 0} \left(\csc x - \dfrac{1}{x} \right)$

30. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x} \right)^x$

31. $\lim_{x \to 0^+} (1 + 2e^x)^{1/x}$

32. $\lim_{x \to 1} \left(\dfrac{1}{x - 1} - \dfrac{x}{\ln x} \right)$

33. $\lim_{x \to 0} (\cos x)^{1/x}$

34. $\lim_{x \to 0^+} (x^{1/2} \ln x)$

35. $\lim_{x \to \infty} e^{\cos x}$

36. $\lim_{x \to \infty} [\ln(x + 1) - \ln(x - 1)]$

37. $\lim_{x \to 0^+} \dfrac{x}{\ln x}$

38. $\lim_{x \to 0^+} (\ln x \cot x)$

39. $\lim_{x \to \infty} \dfrac{\int_1^x \sqrt{1 + e^{-t}}\, dt}{x}$

40. $\lim_{x \to 1^+} \dfrac{\int_1^x \sin t\, dt}{x - 1}$

41. Find each limit. *Hint:* Transform to problems involving a continuous variable x. Assume that $a > 0$.

(a) $\lim_{n \to \infty} \sqrt[n]{a}$

(b) $\lim_{n \to \infty} \sqrt[n]{n}$

(c) $\lim_{n \to \infty} n\left(\sqrt[n]{a} - 1 \right)$

(d) $\lim_{n \to \infty} n\left(\sqrt[n]{n} - 1 \right)$

42. Find each limit.

(a) $\lim_{x\to 0^+} x^x$ (b) $\lim_{x\to 0^+} (x^x)^x$

(c) $\lim_{x\to 0^+} x^{(x^x)}$ (d) $\lim_{x\to 0^+} ((x^x)^x)^x$

(e) $\lim_{x\to 0^+} x^{(x^{(x^x)})}$

43. Graph $y = x^{1/x}$ for $x > 0$. Show what happens for very small x and very large x. Indicate the maximum value.

44. Find each limit.

(a) $\lim_{x\to 0^+} (1^x + 2^x)^{1/x}$ (b) $\lim_{x\to 0^-} (1^x + 2^x)^{1/x}$

(c) $\lim_{x\to \infty} (1^x + 2^x)^{1/x}$ (d) $\lim_{x\to -\infty} (1^x + 2^x)^{1/x}$

45. For $k \geq 0$, find

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}$$

Hint: Though this has the ∞/∞ form, l'Hôpital's Rule is not helpful. Think of a Riemann sum.

46. Let $c_1, c_2, \ldots, c_n$ be positive constants with $\sum_{i=1}^{n} c_i = 1$, and let $x_1, x_2, \ldots, x_n$ be positive numbers. Take natural logarithms and then use l'Hôpital's Rule to show that

$$\lim_{t\to 0^+} \left(\sum_{i=1}^{n} c_i x_i^t \right)^{1/t} = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} = \prod_{i=1}^{n} x_i^{c_i}$$

Here $\prod$ means product; that is, $\prod_{i=1}^{n} a_i$ means $a_1 \cdot a_2 \cdot \cdots \cdot a_n$. In particular, if a, b, x, and y are positive and $a + b = 1$, then

$$\lim_{t\to 0^+} (ax^t + by^t)^{1/t} = x^a y^b$$

47. Verify the last statement in Problem 46 by calculating each of the following.

(a) $\lim_{t\to 0^+} \left(\frac{1}{2}2^t + \frac{1}{2}5^t\right)^{1/t}$ (b) $\lim_{t\to 0^+} \left(\frac{1}{5}2^t + \frac{4}{5}5^t\right)^{1/t}$

(c) $\lim_{t\to 0^+} \left(\frac{1}{10}2^t + \frac{9}{10}5^t\right)^{1/t}$

48. Consider $f(x) = n^2 x e^{-nx}$.

(a) Graph $f(x)$ for $n = 1, 2, 3, 4, 5, 6$ on $[0, 1]$ in the same graph window.

(b) For $x > 0$, find $\lim_{n\to\infty} f(x)$.

(c) Evaluate $\int_0^1 f(x)\,dx$ for $n = 1, 2, 3, 4, 5, 6$.

(d) Guess at $\lim_{n\to\infty} \int_0^1 f(x)\,dx$. Then justify your answer rigorously.

CAS **49.** Find the absolute maximum and minimum points (if they exist) for $f(x) = (x^{25} + x^3 + 2^x)e^{-x}$ on $[0, \infty)$.

Answers to Concepts Review: **1.** $f'(x)/g'(x)$ **2.** $\lim_{x\to a} f(x)/[1/g(x)]$ or $\lim_{x\to a} g(x)/[1/f(x)]$ **3.** $\infty - \infty$, 0^0, ∞^0, 1^∞ **4.** $\ln x$

8.3 Improper Integrals: Infinite Limits of Integration

In the definition of $\int_a^b f(x)\,dx$, it was assumed that the interval $[a, b]$ was finite. However, in many applications in physics, economics, and probability we wish to allow a or b (or both) to be ∞ or $-\infty$. We must therefore find a way to give meaning to symbols like

$$\int_0^\infty \frac{1}{1 + x^2}\,dx, \qquad \int_{-\infty}^{-1} xe^{-x^2}\,dx, \qquad \int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$$

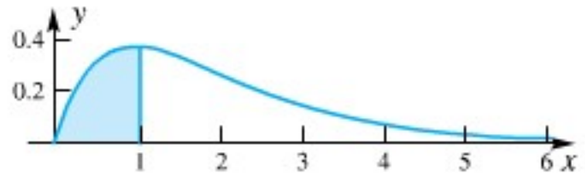
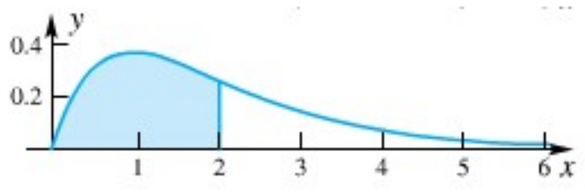
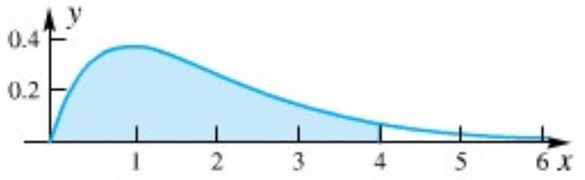
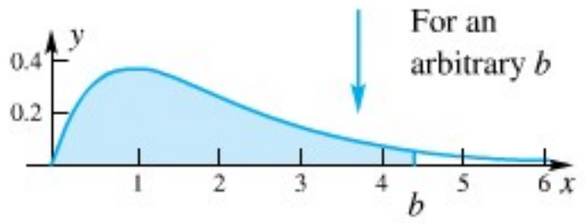
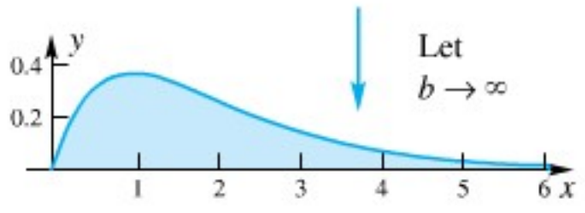
These integrals are called **improper integrals** with infinite limits.

One Infinite Limit Consider the function $f(x) = xe^{-x}$. It makes perfectly good sense to ask for $\int_0^1 xe^{-x}\,dx$ or $\int_0^2 xe^{-x}\,dx$, or indeed for $\int_0^b xe^{-x}\,dx$, where b is any positive number. As the table on the next page indicates, as we increase the upper limit in the definite integral the value of the integral (the area under the curve) increases, but apparently not without bound (in this example, at least). To give meaning to $\int_0^\infty xe^{-x}\,dx$ we begin by integrating from 0 to an arbitrary upper limit, say b, which using integration by parts gives

$$\int_0^b xe^{-x}\,dx = [-xe^{-x}]_0^b - \int_0^b (-e^{-x})\,dx = 1 - e^{-b} - be^{-b}$$

Now, imagine that the value of b marches off to infinity. (See the accompanying table.) As the above calculation shows, if we let $b \to \infty$, the value of the definite integral converges to 1. Thus, it seems natural to define

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_0^b xe^{-x}\,dx = \lim_{b\to\infty} (1 - e^{-b} - be^{-b}) = 1$$

Integral	Picture	Exact Value	Numerical Approximation
$\int_0^1 xe^{-x}\,dx$		$1 - e^{-1} - 1e^{-1}$	0.2642
$\int_0^2 xe^{-x}\,dx$		$1 - e^{-2} - 2e^{-2}$	0.5940
$\int_0^3 xe^{-x}\,dx$		$1 - e^{-3} - 3e^{-3}$	0.8009
$\int_0^b xe^{-x}\,dx$		$1 - e^{-b} - be^{-b}$	
$\int_0^\infty xe^{-x}\,dx$		$\lim_{b\to\infty}\left[1 - e^{-b} - be^{-b}\right] = 1$	

Here is the general definition.

Definition

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

If the limits on the right exist and have finite values, then we say that the corresponding improper integrals **converge** and have those values. Otherwise, the integrals are said to **diverge.**

EXAMPLE 1 Find, if possible, $\displaystyle\int_{-\infty}^{-1} xe^{-x^2}\,dx$.

SOLUTION

$$\int_a^{-1} xe^{-x^2}\,dx = -\frac{1}{2}\int_a^{-1} e^{-x^2}(-2x\,dx) = \left[-\frac{1}{2}e^{-x^2}\right]_a^{-1}$$

$$= -\frac{1}{2}e^{-1} + \frac{1}{2}e^{-a^2}$$

Thus,

$$\int_{-\infty}^{-1} xe^{-x^2}\,dx = \lim_{a\to-\infty}\left[-\frac{1}{2}e^{-1} + \frac{1}{2}e^{-a^2}\right] = -\frac{1}{2e}$$

We say the integral converges and has value $-1/2e$. ■

EXAMPLE 2 Find, if possible, $\int_0^\infty \sin x\,dx$.

SOLUTION

$$\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty}[-\cos x]_0^b$$

$$= \lim_{b\to\infty}[1 - \cos b]$$

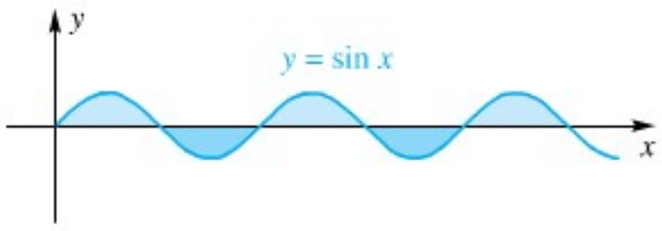


Figure 1

The latter limit does not exist; we conclude that the given integral diverges. Think about the geometric meaning of $\int_0^\infty \sin x\,dx$ to support this result (Figure 1). ■

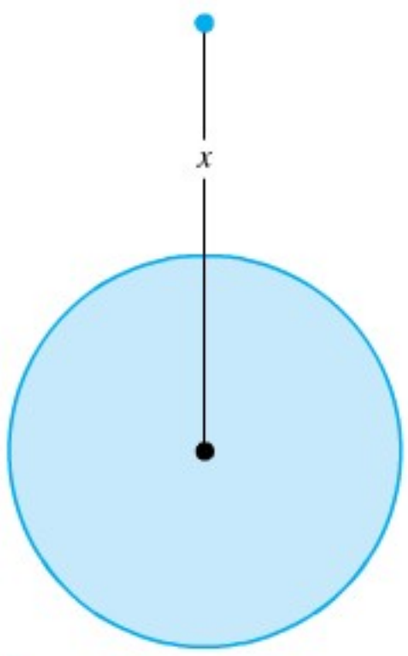


Figure 2

EXAMPLE 3 According to Newton's Inverse Square Law, the force exerted by the earth on a space capsule is $-k/x^2$, where x is the distance (in miles, for instance) from the capsule to the center of the earth (Figure 2). The force $F(x)$ required to lift the capsule is therefore $F(x) = k/x^2$. How much work is done in propelling a 1000-pound capsule out of the earth's gravitational field?

SOLUTION We can evaluate k by noting that at $x = 3960$ miles (the radius of the earth) $F = 1000$ pounds. This yields $k = 1000(3960)^2 \approx 1.568 \times 10^{10}$. The work done in mile-pounds is therefore

$$1.568\times10^{10}\int_{3960}^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} 1.568\times10^{10}\left[-\frac{1}{x}\right]_{3960}^b$$

$$= \lim_{b\to\infty} 1.568\times10^{10}\left[-\frac{1}{b} + \frac{1}{3960}\right]$$

$$= \frac{1.568\times10^{10}}{3960} \approx 3.96\times10^6$$ ■

Both Limits Infinite We can now give a definition for $\int_{-\infty}^\infty f(x)\,dx$.

Definition

If both $\int_{-\infty}^0 f(x)\,dx$ and $\int_0^\infty f(x)\,dx$ converge, then $\int_{-\infty}^\infty f(x)\,dx$ is said to converge and have value

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

Otherwise, $\int_{-\infty}^\infty f(x)\,dx$ diverges.

EXAMPLE 4 Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$ or state that it diverges.

SOLUTION

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty}[\tan^{-1}x]_0^b$$

$$= \lim_{b\to\infty}[\tan^{-1}b - \tan^{-1}0] = \frac{\pi}{2}$$

Since the integrand is an even function,

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad \blacksquare$$

We will use the notation $[F(x)]_a^{\infty}$ to mean $\lim_{b\to\infty} F(b) - F(a)$. Similar definitions apply to $[F(x)]_{-\infty}^{a}$ and to $[F(x)]_{-\infty}^{\infty}$. Note that in none of these cases are we "substituting" infinity. Each is defined as a limit, which agrees with our approach to determining improper integrals.

Probability Density Functions When we first introduced random variables and probability density functions back in Section 5.7 we had to restrict attention to cases where the set of possible outcomes was bounded. In many situations, there is no upper (or lower) limit for the set of possible outcomes. For example, there is no upper bound on how long a battery will last, or how strong a mix of concrete is. Now that we have covered improper integrals, we can dispense with this restriction.

If the PDF $f(x)$ of a continuous random variable X is defined to be 0 outside the set of possible outcomes, then the requirements for a PDF are

1. $f(x) \geq 0$
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

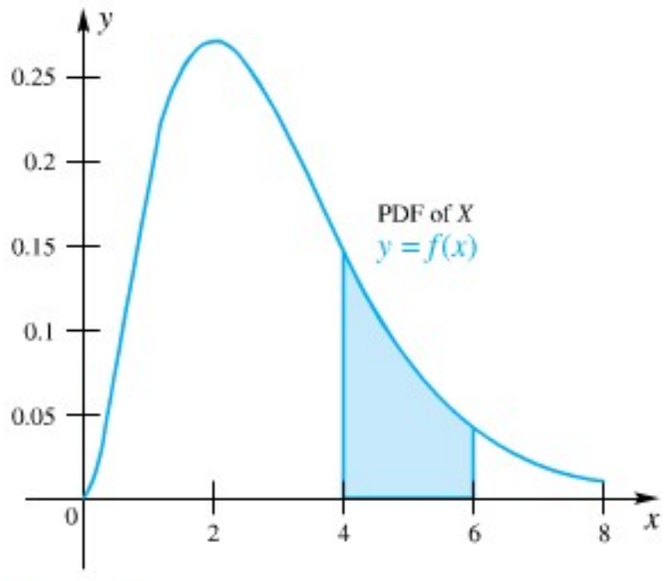


Figure 3

The PDF of a random variable allows us to find probabilities by integration; for example, Figure 3 illustrates the probability that X is between 4 and 6.

The mean and variance of a random variable are then defined by

$$\mu = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

The variance σ^2 of a random variable is a measure of the dispersion, or "spread-out-ness" of the probability, and it can be computed from (see Problem 41 of Section 5.7)

$$\sigma^2 = E(X^2) - \mu^2$$

When σ^2 is small, the distribution of probability is, roughly speaking, clustered very closely around the mean; when σ^2 is large, the probability is more spread out.

The next two examples, and some of the exercises, introduce several useful families of probability distributions.

EXAMPLE 5 The **exponential distribution,** which is sometimes used to model the lifetimes of electrical or mechanical components, has PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } 0 \leq x \\ 0, & \text{otherwise} \end{cases}$$

where λ is some positive constant.

(a) Show that this is a valid PDF.
(b) Find the mean μ and the variance σ^2.
(c) Find the cumulative distribution function (CDF) $F(x)$.
(d) If a component's lifetime X measured in hours is a random variable having an exponential distribution with $\lambda = 0.01$, what is the probability that the component works for at least 20 hours?

SOLUTION

(a) The function f is always nonnegative and

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx \\ &= 0 + [-e^{-\lambda x}]_0^{\infty} \\ &= 1\end{aligned}$$

so $f(x)$ is a valid PDF.

(b)

$$\begin{aligned}E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx \\ &= \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x\lambda e^{-\lambda x}\,dx\end{aligned}$$

We apply integration by parts in the second integral: $u = x$, $dv = \lambda e^{-\lambda x}\,dx$, so that $du = dx$, $v = -e^{-\lambda x}$. Thus

$$\begin{aligned}E(X) &= [-x\lambda e^{-\lambda x}]_0^{\infty} - \int_{0}^{\infty} (-e^{-\lambda x})\,dx \\ &= (-0 + 0) + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty} \\ &= \frac{1}{\lambda}\end{aligned}$$

The variance is

$$\begin{aligned}\sigma^2 &= E(X^2) - \mu^2 \\ &= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\frac{1}{\lambda}\right)^2 \\ &= \int_{-\infty}^{0} x^2 \cdot 0\,dx + \int_{0}^{\infty} x^2 \lambda e^{-\lambda x}\,dx - \frac{1}{\lambda^2} \\ &= [-x^2 e^{-\lambda x}]_0^{\infty} - \int_{0}^{\infty} (-e^{-\lambda x})\,2x\,dx - \frac{1}{\lambda^2} \\ &= (-0 + 0) + 2\int_{0}^{\infty} xe^{-\lambda x}\,dx - \frac{1}{\lambda^2} \\ &= 2\,\frac{1}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}\end{aligned}$$

(c) For $x < 0$, the CDF is $F(x) = P(X \le x) = 0$. For $x \ge 0$,

$$\begin{aligned}F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \lambda e^{-\lambda t}\,dt \\ &= 0 + [-e^{-\lambda t}]_0^{x} \\ &= 1 - e^{-\lambda x}\end{aligned}$$

Figure 4

A graph of the CDF is shown in Figure 4.

(d) Set $\lambda = 0.01$. The probability that the component works for at least 20 hours is the probability that the lifetime is 20 hours or greater:

$$\begin{aligned} P(X > 20) &= \int_{20}^{\infty} 0.01e^{-0.01x}\,dx \\ &= [-e^{-0.01x}]_{20}^{\infty} \\ &= 0 - (-e^{-0.01\cdot 20}) \\ &= e^{-0.2} \\ &\approx 0.819 \end{aligned}$$

■

The **normal distribution** is the familiar bell-shaped curve. It is really a family of distributions since the mean μ can be any number and the variance can be any positive number σ^2. The normal distribution with parameters μ and σ^2 has PDF

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp[-(x-\mu)^2/2\sigma^2]$$

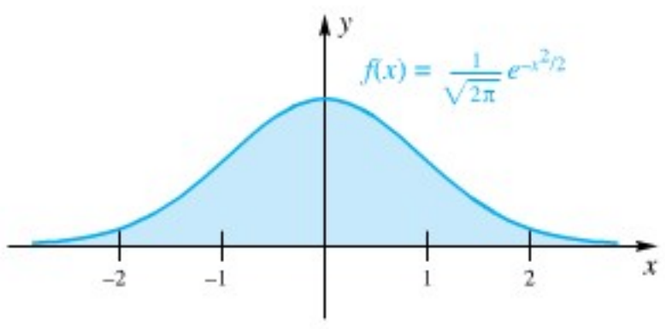


Figure 5

(The parameters μ and σ^2 turn out to be equal to the mean and variance, respectively, thus justifying the use of the Greek letters μ and σ.) Figure 5 shows a plot of the PDF for the normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$. It is surprisingly difficult to show that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp[-(x-\mu)^2/2\sigma^2]\,dx = 1$$

although we will do so later (Section 13.4). Other properties of the normal distribution include the following:

(a) its graph is symmetric about the line $x = \mu$;
(b) it has a maximum at $x = \mu$;
(c) it has inflection points when $x = \mu \pm \sigma$;
(d) the mean is μ;
(e) the variance is σ^2.

Problem 33 involves some other properties of the normal PDF. The normal distribution with $\mu = 0$ and $\sigma^2 = 1$ is called the **standard normal distribution**. This is the normal distribution that is graphed in Figure 5.

EXAMPLE 6 Show that

(a) $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} xe^{-x^2/2}\,dx = 0$

(b) $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} x^2e^{-x^2/2}\,dx = 1$

SOLUTION

(a) $$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} xe^{-x^2/2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\sqrt{2\pi}} \int_{0}^{b} e^{-x^2/2}(-x\,dx)\right]$$

$$= \lim_{b\to\infty}\left[-\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right]_0^b$$

$$= \frac{1}{\sqrt{2\pi}}$$

Since $xe^{-x^2/2}$ is an odd function,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} xe^{-x^2/2}\,dx = -\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-x^2/2}\,dx = -\frac{1}{\sqrt{2\pi}}$$

Thus,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} xe^{-x^2/2}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-x^2/2}\,dx$$

$$= -\frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} = 0$$

(b) Since $e^{-x^2/2}$ is an even function and since $\displaystyle\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = 1$,

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x^2/2}\,dx = \frac{1}{2}$$

We then apply integration by parts and l'Hôpital's Rule.

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} x^2e^{-x^2/2}\,dx = \lim_{b\to\infty}\frac{1}{\sqrt{2\pi}}\int_{0}^{b}(x)(e^{-x^2/2}x)\,dx$$

$$= \lim_{b\to\infty}\frac{1}{\sqrt{2\pi}}\left(\left[-xe^{-x^2/2}\right]_0^b + \int_0^b e^{-x^2/2}\,dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(0 + \int_0^{\infty} e^{-x^2/2}\,dx\right) = \frac{1}{2}$$

Since $x^2e^{-x^2/2}$ is an even function, we get the same contribution to the left of zero, and so

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2e^{-x^2/2}\,dx = \frac{1}{2} + \frac{1}{2} = 1$$ ■

Figure 6

The Paradox of Gabriel's Horn Let the curve $y = 1/x$ on $[1, \infty)$ be revolved about the x-axis, thereby generating a surface called Gabriel's horn (Figure 6). We claim that

1. the volume V of this horn is finite;
2. the surface area A of the horn is infinite.

To put the results in practical terms, they seem to say that the horn can be filled with a finite amount of paint, and yet there is not enough to paint its inside surface. Before we try to unravel this paradox, let us establish (1) and (2). We use results for volume from Section 5.2 and for surface area from Section 5.4.

Hans Memling (1425/40–1494). Last Judgement, detail of right panel: angel blowing a trumpet and the damned falling into Hell. Pomorskie Museum, Gdansk, Poland. Scala/Art Resource, N.Y.

$$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \pi \int_1^b x^{-2}\, dx$$

$$= \lim_{b\to\infty}\left[-\frac{\pi}{x}\right]_1^b = \pi$$

$$A = \int_1^\infty 2\pi y\, ds = \int_1^\infty 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx$$

$$= \lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx$$

Now,

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x}$$

Thus,

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx > \int_1^b \frac{1}{x}\, dx = \ln b$$

and since $\ln b \to \infty$ as $b \to \infty$, we conclude that A is infinite.

Is something wrong with our mathematics? No. Imagine the horn to be slit along the side, opened up, and laid flat. Given a finite amount of paint, we could not possibly paint this surface with a paint coat of *uniform* thickness. However, we could do it if we allow the paint coat to get thinner and thinner as we move farther and farther from the horn's fat end. And, of course, that is exactly what happens when we fill the unslit horn with π cubic units of paint. (Imaginary paint can be spread to arbitrary thinness.)

This problem involved the study of two integrals of the form $\int_1^\infty 1/x^p\, dx$. For later reference, we now analyze this integral for all values of p.

Gabriel Paves a Street

When told to pave the infinite street $0 \le x < \infty, 0 \le y \le 1$ with pure gold, Gabriel obeyed but made the thickness h of the gold at x satisfy

$$h = e^{-x}$$

How much gold did it take?

$$V = \int_0^\infty e^{-x}\, dx = \lim_{b\to\infty} \int_0^b e^{-x}\, dx$$

$$= \lim_{b\to\infty} [-e^{-x}]_0^b = 1$$

Just 1 cubic unit.

EXAMPLE 7 Show that $\int_1^\infty 1/x^p\, dx$ diverges for $p \le 1$ and converges for $p > 1$.

SOLUTION We showed in our solution to Gabriel's horn that the integral diverges for $p = 1$. If $p \ne 1$,

$$\int_1^\infty \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b x^{-p}\, dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\right]\left[\frac{1}{b^{p-1}} - 1\right] = \begin{cases} \infty & \text{if } p < 1 \\ \dfrac{1}{p-1} & \text{if } p > 1 \end{cases}$$

The conclusion follows. ■

Concepts Review

1. $\int_a^\infty f(x)\, dx$ is said to _____ if $\lim_{b\to\infty} \int_a^b f(x)\, dx$ exists and is finite.

2. $\int_0^\infty \cos x\, dx$ does not converge because _____ does not exist.

3. $\int_{-\infty}^\infty f(x)\, dx$ is said to diverge if either _____ or _____ diverges.

4. $\int_1^\infty (1/x^p)\, dx$ converges if and only if _____.

Problem Set 8.3

In Problems 1–24, evaluate each improper integral or show that it diverges.

1. $\displaystyle\int_{100}^{\infty} e^x\,dx$

2. $\displaystyle\int_{-\infty}^{-5} \frac{dx}{x^4}$

3. $\displaystyle\int_{1}^{\infty} 2xe^{-x^2}\,dx$

4. $\displaystyle\int_{-\infty}^{1} e^{4x}\,dx$

5. $\displaystyle\int_{9}^{\infty} \frac{x\,dx}{\sqrt{1+x^2}}$

6. $\displaystyle\int_{1}^{\infty} \frac{dx}{\sqrt{\pi x}}$

7. $\displaystyle\int_{1}^{\infty} \frac{dx}{x^{1.00001}}$

8. $\displaystyle\int_{10}^{\infty} \frac{x}{1+x^2}\,dx$

9. $\displaystyle\int_{1}^{\infty} \frac{dx}{x^{0.99999}}$

10. $\displaystyle\int_{1}^{\infty} \frac{x}{(1+x^2)^2}\,dx$

11. $\displaystyle\int_{e}^{\infty} \frac{1}{x\ln x}\,dx$

12. $\displaystyle\int_{e}^{\infty} \frac{\ln x}{x}\,dx$

13. $\displaystyle\int_{2}^{\infty} \frac{\ln x}{x^2}\,dx$

14. $\displaystyle\int_{1}^{\infty} xe^{-x}\,dx$

15. $\displaystyle\int_{-\infty}^{1} \frac{dx}{(2x-3)^3}$

16. $\displaystyle\int_{4}^{\infty} \frac{dx}{(\pi-x)^{2/3}}$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+9}}\,dx$

18. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+16)^2}$

19. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10}\,dx$

20. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{e^{2|x|}}\,dx$

21. $\displaystyle\int_{-\infty}^{\infty} \operatorname{sech} x\,dx$ *Hint:* Use a table of integrals or a CAS.

22. $\displaystyle\int_{1}^{\infty} \operatorname{csch} x\,dx$

23. $\displaystyle\int_{0}^{\infty} e^{-x}\cos x\,dx$ *Hint:* Use table of integrals or a CAS.

24. $\displaystyle\int_{0}^{\infty} e^{-x}\sin x\,dx$

25. Find the area of the region under the curve $y = 2/(4x^2-1)$ to the right of $x = 1$. *Hint:* Use partial fractions.

26. Find the area of the region under the curve $y = 1/(x^2+x)$ to the right of $x = 1$.

27. Suppose that Newton's law for the force of gravity had the form $-k/x$ rather than $-k/x^2$ (see Example 3). Show that it would then be impossible to send anything out of the earth's gravitational field.

28. If a 1000-pound capsule weighs only 165 pounds on the moon (radius 1080 miles), how much work is done in propelling this capsule out of the moon's gravitational field (see Example 3)?

29. Suppose that a company expects its annual profits t years from now to be $f(t)$ dollars and that interest is considered to be compounded continuously at an annual rate r. Then the present value of all future profits can be shown to be

$$FP = \int_{0}^{\infty} e^{-rt} f(t)\,dt$$

Find FP if $r = 0.08$ and $f(t) = 100{,}000$.

30. Do Problem 29 assuming that $f(t) = 100{,}000 + 1000t$.

31. A continuous random variable X has a **uniform distribution** if it has a probability density function of the form

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{if } x \le a \text{ or } x \ge b \end{cases}$$

(a) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

(b) Find the mean μ and variance σ^2 of the uniform distribution.

(c) If $a = 0$ and $b = 10$, find the probability that X is less than 2.

32. A random variable X has a **Weibull distribution** if it has probability density function

$$f(x) = \begin{cases} \dfrac{\beta}{\theta}\left(\dfrac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^{\beta}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

(a) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$. (Assume $\beta > 1$.)

(b) If $\theta = 3$ and $\beta = 2$, find the mean μ and the variance σ^2.

(c) If the lifetime of a computer monitor is a random variable X that has a Weibull distribution with $\theta = 3$ and $\beta = 2$ (where age is measured in years) find the probability that a monitor fails before two years.

33. Sketch the graph of the normal probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2}$$

and show, using calculus, that σ is the distance from the mean μ to the x-coordinate of one of the inflection points.

CAS **34.** The Pareto probability density function has the form

$$f(x) = \begin{cases} \dfrac{CM^k}{x^{k+1}} & \text{if } x \ge M, \\ 0 & \text{if } x < M \end{cases}$$

where k and M are positive constants.

(a) Find the value of C that makes $f(x)$ a probability density function.

(b) For the value of C found in part (a), find the value of the mean μ. Is the mean finite for all positive k? If not, how does the mean depend on k?

(c) For the value of C found in part (a), find the variance σ^2. How does the variance depend on k?

35. The Pareto distribution is often used to model income distribution. Suppose that in some economy the income distribution does follow a Pareto distribution with $k = 3$. Suppose that the mean income is \$20,000.

(a) Find M and C.

(b) Find the variance σ^2.

(c) Find the fraction of income earners who earn more than \$100,000. (*Note:* This is the same as asking what is the probability that a randomly chosen person has an income of more than \$100,000.)

36. In electromagnetic theory, the magnetic potential u at a point on the axis of a circular coil is given by

$$u = Ar \int_a^\infty \frac{dx}{(r^2 + x^2)^{3/2}}$$

where A, r, and a are constants. Evaluate u.

37. There is a subtlety in the definition of $\int_{-\infty}^{\infty} f(x)\,dx$ that is illustrated by the following: Show that

(a) $\int_{-\infty}^{\infty} \sin x\,dx$ diverges and

(b) $\lim_{a\to\infty} \int_{-a}^{a} \sin x\,dx = 0$.

38. Consider an infinitely long wire coinciding with the positive x-axis and having mass density $\delta(x) = (1 + x^2)^{-1}$, $0 \le x < \infty$.

(a) Calculate the total mass of the wire.

(b) Show that this wire does not have a center of mass.

39. Give an example of a region in the first quadrant that gives a solid of finite volume when revolved about the x-axis, but gives a solid of infinite volume when revolved about the y-axis.

40. Let f be a nonnegative continuous function defined on $0 \le x < \infty$ with $\int_0^\infty f(x)\,dx < \infty$. Show that

(a) if $\lim_{x\to\infty} f(x)$ exists it must be 0;

(b) it is possible that $\lim_{x\to\infty} f(x)$ does not exist.

CAS **41.** We can use a computer to approximate $\int_1^\infty f(x)\,dx$ by taking b very large in $\int_1^b f(x)\,dx$ *provided* we know that the first integral converges. Calculate $\int_1^{100} (1/x^p)\,dx$ for $p = 2, 1.1, 1.01, 1$, and 0.99. Note that this gives no hint that the integral $\int_1^\infty (1/x^p)\,dx$ converges for $p > 1$ and diverges for $p \le 1$.

CAS **42.** Approximate $\int_0^a \frac{1}{\pi}(1 + x^2)^{-1}\,dx$ for $a = 10, 50$, and 100.

CAS **43.** Approximate $\int_{-a}^{a} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\,dx$ for $a = 1, 2, 3$, and 4.

Answers to Concepts Review: **1.** converge **2.** $\lim_{b\to\infty} \int_0^b \cos x\,dx$ **3.** $\int_{-\infty}^0 f(x)\,dx$; $\int_0^\infty f(x)\,dx$ **4.** $p > 1$

8.4 Improper Integrals: Infinite Integrands

Considering the many complicated integrations that we have done, here is one that looks simple enough but is incorrect.

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-2}^{1} = -1 - \frac{1}{2} = -\frac{3}{2} \qquad \text{Wrong}$$

One glance at Figure 1 tells us that something is terribly wrong. The value of the integral (if there is one) has to be a positive number. (Why?)

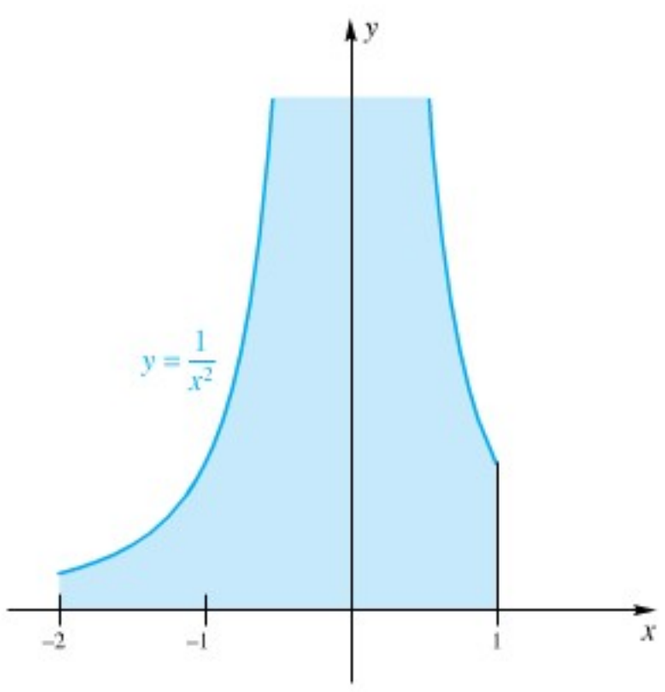


Figure 1

Where is our mistake? To answer, we refer back to Section 4.2. Recall that for a function to be integrable in the standard (or proper) sense it must be bounded. Our function, $f(x) = 1/x^2$, is not bounded, so it is not integrable in the proper sense. We say that $\int_{-2}^{1} x^{-2}\,dx$ is an improper integral with an infinite integrand (*unbounded integrand* is a more accurate but less colorful term).

Until now, we have carefully avoided infinite integrands in all our examples and problems. We could continue to do this, but this would be to avoid a kind of integral that has important applications. Our task for this section is to define and analyze this new kind of integral.

Integrands That Are Infinite at an End Point We give the definition for the case where f tends to infinity at the right end point of the interval of integration. There is a completely analogous definition for the case where f tends to infinity at the left end point.

Definition

Let f be continuous on the half-open interval $[a, b)$ and suppose that $\lim_{x\to b^-} |f(x)| = \infty$. Then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided that this limit exists and is finite, in which case we say that the integral converges. Otherwise, we say that the integral diverges.

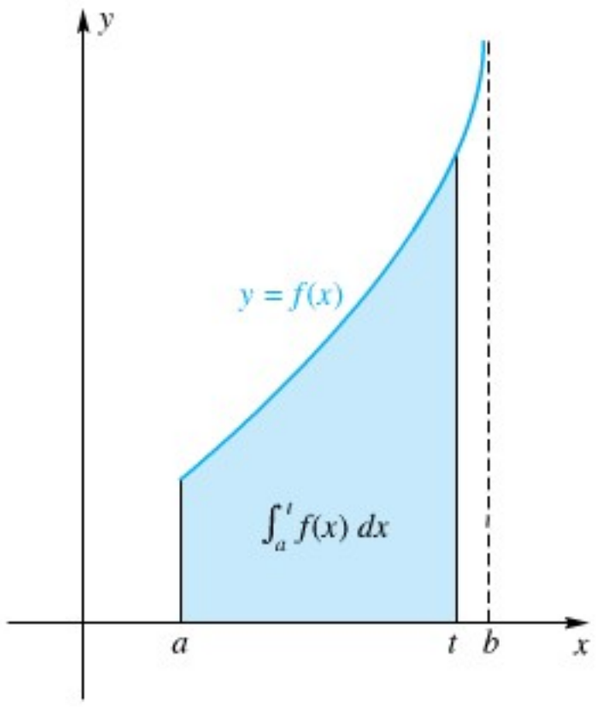


Figure 2

Note the geometric interpretation shown in Figure 2.

EXAMPLE 1 Evaluate, if possible, the improper integral $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}}$.

SOLUTION Note that the integrand tends to infinity at 2.

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{t\to 2^-}\int_0^t \frac{dx}{\sqrt{4-x^2}} = \lim_{t\to 2^-}\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^t$$

$$= \lim_{t\to 2^-}\left[\sin^{-1}\left(\frac{t}{2}\right) - \sin^{-1}\left(\frac{0}{2}\right)\right] = \frac{\pi}{2}$$

■

EXAMPLE 2 Evaluate, if possible, $\displaystyle\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx$.

SOLUTION

$$\int_0^{16} x^{-1/4}\,dx = \lim_{t\to 0^+}\int_t^{16} x^{-1/4}\,dx = \lim_{t\to 0^+}\left[\frac{4}{3}x^{3/4}\right]_t^{16}$$

$$= \lim_{t\to 0^+}\left[\frac{32}{3} - \frac{4}{3}t^{3/4}\right] = \frac{32}{3}$$

■

Two Key Examples

From Example 7 of Section 8.3, we learned that

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if and only if $p > 1$. From Example 4 of the present section, we learn that

$$\int_0^1 \frac{1}{x^p}\,dx$$

converges if and only if $p < 1$. The first has an infinite limit of integration; the second has an infinite integrand. If you feel at home with these two integrals, you should also be at ease with any other improper integrals that you may meet.

EXAMPLE 3 Evaluate, if possible, $\displaystyle\int_0^1 \frac{1}{x}\,dx$.

SOLUTION

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}[\ln x]_t^1$$

$$= \lim_{t\to 0^+}[-\ln t] = \infty$$

We conclude that the integral diverges. ■

EXAMPLE 4 Show that $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$, but diverges if $p \geq 1$.

SOLUTION Example 3 took care of the case $p = 1$. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-p}\,dx = \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1$$

$$= \lim_{t\to 0^+}\left[\frac{1}{1-p} - \frac{1}{1-p}\cdot\frac{1}{t^{p-1}}\right] = \begin{cases} \dfrac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p > 1 \end{cases}$$

■

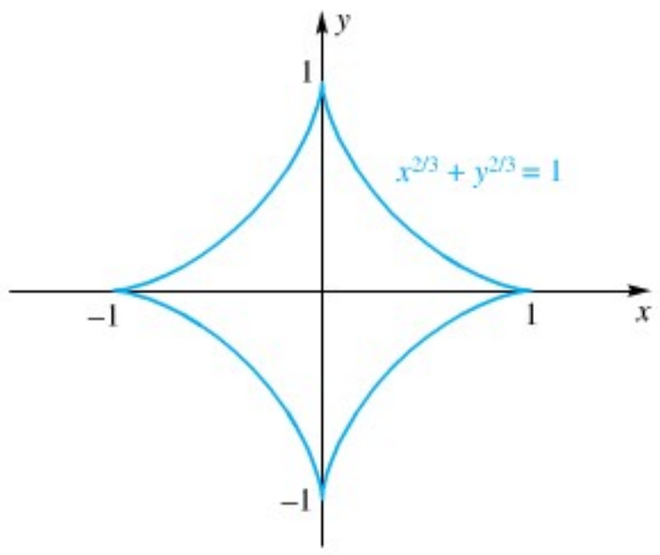


Figure 3

EXAMPLE 5 Sketch the graph of the hypocycloid of four cusps, $x^{2/3} + y^{2/3} = 1$, and find its perimeter.

≈ **SOLUTION** The graph is shown in Figure 3. To find the perimeter, it is enough to find the length L of the first quadrant portion and quadruple it. We estimate L to be a bit more than $\sqrt{2} \approx 1.4$. Its exact value (see Section 5.4) is

$$L = \int_0^1 \sqrt{1 + (y')^2}\,dx$$

By implicit differentiation of $x^{2/3} + y^{2/3} = 1$, we obtain

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

or

$$y' = -\frac{y^{1/3}}{x^{1/3}}$$

Thus,

$$1 + (y')^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = 1 + \frac{1 - x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}}$$

and so

$$L = \int_0^1 \sqrt{1 + (y')^2}\, dx = \int_0^1 \frac{1}{x^{1/3}}\, dx$$

The value of this improper integral can be read from the solution to Example 4; it is $L = 1/\left(1 - \frac{1}{3}\right) = \frac{3}{2}$. We conclude that the hypocycloid has perimeter $4L = 6$. ■

Integrands That Are Infinite at an Interior Point The integral $\int_{-2}^{1} 1/x^2\, dx$ of our introduction has an integrand that tends to infinity at $x = 0$, an interior point of the interval $[-2, 1]$. Here is the appropriate definition to give meaning to such an integral.

Definition

Let f be continuous on $[a, b]$ except at a number c, where $a < c < b$, and suppose that $\lim_{x \to c} |f(x)| = \infty$. Then we define

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

provided both integrals on the right converge. Otherwise, we say that $\int_a^b f(x)\, dx$ diverges.

EXAMPLE 6 Show that $\int_{-2}^{1} 1/x^2\, dx$ diverges.

SOLUTION

$$\int_{-2}^{1} \frac{1}{x^2}\, dx = \int_{-2}^{0} \frac{1}{x^2}\, dx + \int_0^1 \frac{1}{x^2}\, dx$$

The second of the integrals on the right diverges by Example 4. This is enough to give the conclusion. ■

EXAMPLE 7 Evaluate, if possible, the improper integral $\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

SOLUTION The integrand tends to infinity at $x = 1$ (see Figure 4). Thus,

$$\begin{aligned}
\int_0^3 \frac{dx}{(x-1)^{2/3}} &= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{t \to 1^-} \int_0^t \frac{dx}{(x-1)^{2/3}} + \lim_{s \to 1^+} \int_s^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{t \to 1^-} [3(x-1)^{1/3}]_0^t + \lim_{s \to 1^+} [3(x-1)^{1/3}]_s^3 \\
&= 3 \lim_{t \to 1^-} [(t-1)^{1/3} + 1] + 3 \lim_{s \to 1^+} [2^{1/3} - (s-1)^{1/3}] \\
&= 3 + 3(2^{1/3}) \approx 6.78
\end{aligned}$$

■

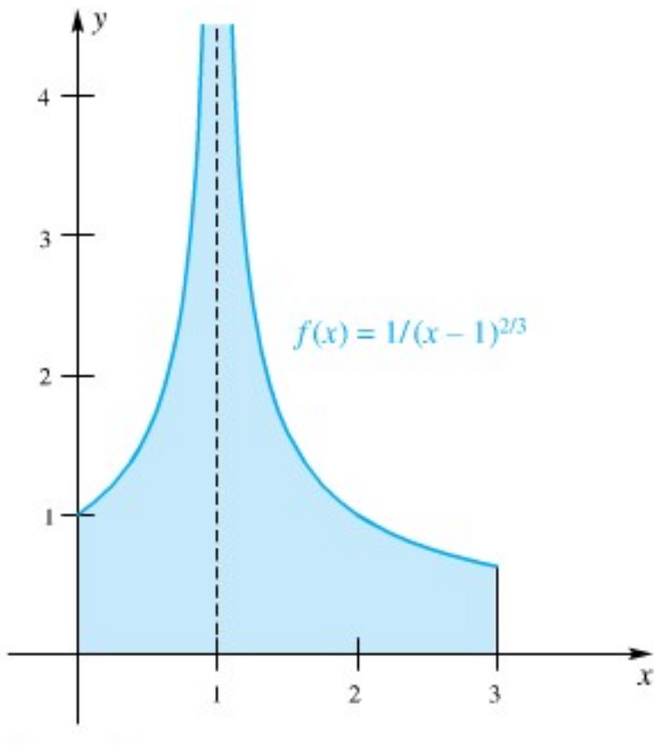


Figure 4

Concepts Review

1. The integral $\int_0^1 (1/\sqrt{x})\,dx$ does not exist in the proper sense because the function $f(x) = 1/\sqrt{x}$ is _____ on the interval $(0, 1]$.

2. Considered as an improper integral, $\int_0^1 (1/\sqrt{x})\,dx = \lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx =$ _____.

3. The improper integral $\int_0^4 \left(1/\sqrt{4-x}\right)dx$ is defined by _____.

4. The improper integral $\int_0^1 (1/x^p)\,dx$ converges if and only if _____.

Problem Set 8.4

In Problems 1–32, evaluate each improper integral or show that it diverges.

1. $\displaystyle\int_1^3 \frac{dx}{(x-1)^{1/3}}$

2. $\displaystyle\int_1^3 \frac{dx}{(x-1)^{4/3}}$

3. $\displaystyle\int_3^{10} \frac{dx}{\sqrt{x-3}}$

4. $\displaystyle\int_0^9 \frac{dx}{\sqrt{9-x}}$

5. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

6. $\displaystyle\int_{100}^{\infty} \frac{x}{\sqrt{1+x^2}}\,dx$

7. $\displaystyle\int_{-1}^3 \frac{1}{x^3}\,dx$

8. $\displaystyle\int_5^{-5} \frac{1}{x^{2/3}}\,dx$

9. $\displaystyle\int_{-1}^{128} x^{-5/7}\,dx$

10. $\displaystyle\int_0^1 \frac{x}{\sqrt[3]{1-x^2}}\,dx$

11. $\displaystyle\int_0^4 \frac{dx}{(2-3x)^{1/3}}$

12. $\displaystyle\int_{\sqrt{5}}^{\sqrt{8}} \frac{x}{(16-2x^2)^{2/3}}\,dx$

13. $\displaystyle\int_0^{-4} \frac{x}{16-2x^2}\,dx$

14. $\displaystyle\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$

15. $\displaystyle\int_{-2}^{-1} \frac{dx}{(x+1)^{4/3}}$

16. $\displaystyle\int_0^3 \frac{dx}{x^2+x-2}$

17. $\displaystyle\int_0^3 \frac{dx}{x^3-x^2-x+1}$

18. $\displaystyle\int_0^{27} \frac{x^{1/3}}{x^{2/3}-9}\,dx$

19. $\displaystyle\int_0^{\pi/4} \tan 2x\,dx$

20. $\displaystyle\int_0^{\pi/2} \csc x\,dx$

21. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1-\cos x}\,dx$

22. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}}\,dx$

23. $\displaystyle\int_0^{\pi/2} \tan^2 x \sec^2 x\,dx$

24. $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{(\tan x - 1)^2}\,dx$

25. $\displaystyle\int_0^{\pi} \frac{dx}{\cos x - 1}$

26. $\displaystyle\int_{-3}^{-1} \frac{dx}{x\sqrt{\ln(-x)}}$

27. $\displaystyle\int_0^{\ln 3} \frac{e^x\,dx}{\sqrt{e^x-1}}$

28. $\displaystyle\int_2^4 \frac{dx}{\sqrt{4x-x^2}}$

29. $\displaystyle\int_1^e \frac{dx}{x\ln x}$

30. $\displaystyle\int_1^{10} \frac{dx}{x\ln^{100} x}$

31. $\displaystyle\int_{2c}^{4c} \frac{dx}{\sqrt{x^2-4c^2}}$

32. $\displaystyle\int_c^{2c} \frac{x\,dx}{\sqrt{x^2+xc-2c^2}}, c > 0$

33. It is often possible to change an improper integral into a proper one by using integration by parts. Consider $\lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt{x}(1+x)}$. Use integration by parts on the interval $[c, 1]$ where $0 < c < 1$ to show that

$$\int_c^1 \frac{dx}{\sqrt{x}(1+x)} = 1 - \frac{2\sqrt{c}}{c+1} + 2\int_c^1 \frac{\sqrt{x}\,dx}{(1+x)^2}$$

and thus conclude that upon taking the limit as $c \to 0$ an improper integral can be turned into a proper integral.

34. Use integration by parts and the technique of Problem 33 to transform the improper integral $\int_0^1 \frac{dx}{\sqrt{x(1+x)}}$ into a proper integral.

35. If $f(x)$ tends to infinity at both a and b, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where c is any point between a and b, provided of course that both the latter integrals converge. Otherwise, we say that the given integral diverges. Use this to evaluate $\int_{-3}^3 \frac{x}{\sqrt{9-x^2}}\,dx$ or show that it diverges.

36. Evaluate $\int_{-3}^3 \frac{x}{9-x^2}\,dx$ or show that it diverges. See Problem 35.

37. Evaluate $\int_{-4}^4 \frac{1}{16-x^2}\,dx$ or show that it diverges. See Problem 35.

38. Evaluate $\int_{-1}^1 \frac{1}{x\sqrt{-\ln|x|}}\,dx$ or show that it diverges.

39. If $\lim_{x\to 0^+} f(x) = \infty$, we define

$$\int_0^{\infty} f(x)\,dx = \lim_{c\to 0^+} \int_c^1 f(x)\,dx + \lim_{b\to\infty} \int_1^b f(x)\,dx$$

provided both limits exist. Otherwise, we say that $\int_0^{\infty} f(x)\,dx$ diverges. Show that $\int_0^{\infty} \frac{1}{x^p}\,dx$ diverges for all p.

40. Suppose that f is continuous on $[0, \infty)$ except at $x = 1$, where $\lim_{x\to 1} |f(x)| = \infty$. How would you define $\int_0^{\infty} f(x)\,dx$?

41. Find the area of the region between the curves $y = (x - 8)^{-2/3}$ and $y = 0$ for $0 \le x < 8$.

42. Find the area of the region between the curves $y = 1/x$ and $y = 1/(x^3 + x)$ for $0 < x \le 1$.

43. Let R be the region in the first quadrant below the curve $y = x^{-2/3}$ and to the left of $x = 1$.

(a) Show that the area of R is finite by finding its value.

(b) Show that the volume of the solid generated by revolving R about the x-axis is infinite.

44. Find b so that $\displaystyle\int_0^b \ln x\, dx = 0$.

45. Is $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$ an improper integral? Explain.

EXPL **46. (Comparison Test)** If $0 \le f(x) \le g(x)$ on $[a, \infty)$, it can be shown that the convergence of $\displaystyle\int_a^\infty g(x)\, dx$ implies the convergence of $\displaystyle\int_a^\infty f(x)\, dx$, and the divergence of $\displaystyle\int_a^\infty f(x)\, dx$ implies the divergence of $\displaystyle\int_a^\infty g(x)\, dx$. Use this to show that $\displaystyle\int_1^\infty \frac{1}{x^4(1 + x^4)}\, dx$ converges.

Hint: On $[1, \infty)$, $1/[x^4(1 + x^4)] \le 1/x^4$.

47. Use the Comparison Test of Problem 46 to show that $\displaystyle\int_1^\infty e^{-x^2}\, dx$ converges. *Hint:* $e^{-x^2} \le e^{-x}$ on $[1, \infty)$.

48. Use the Comparison Test of Problem 46 to show that $\displaystyle\int_2^\infty \frac{1}{\sqrt{x + 2} - 1}\, dx$ diverges.

49. Use the Comparison Test of Problem 46 to determine whether $\displaystyle\int_1^\infty \frac{1}{x^2 \ln(x + 1)}\, dx$ converges or diverges.

50. Formulate a comparison test for improper integrals with infinite integrands.

51. (a) Use Example 2 of Section 8.2 to show that for any positive number n there is a number M such that

$$0 < \frac{x^{n-1}}{e^x} \le \frac{1}{x^2} \text{ for } x \ge M$$

(b) Use part (a) and Problem 46 to show that $\displaystyle\int_1^\infty x^{n-1}e^{-x}\, dx$ converges.

52. Using Problem 50, prove that $\displaystyle\int_0^1 x^{n-1}e^{-x}\, dx$ converges for $n > 0$.

EXPL **53. (Gamma Function)** Let $\Gamma(n) = \displaystyle\int_0^\infty x^{n-1}e^{-x}\, dx$, $n > 0$. This integral converges by Problems 51 and 52. Show each of the following (note that the gamma function is defined for every positive real number n):

(a) $\Gamma(1) = 1$ (b) $\Gamma(n + 1) = n\Gamma(n)$

(c) $\Gamma(n + 1) = n!$, if n is a positive integer.

CAS **54.** Evaluate $\displaystyle\int_0^\infty x^{n-1}e^{-x}\, dx$ for $n = 1, 2, 3, 4$, and 5, thereby confirming Problem 53(c).

55. The **gamma** probability density function is

$$f(x) = \begin{cases} Cx^{\alpha-1}e^{-\beta x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

where α and β are positive constants. (Both the gamma and the Weibull distributions are used to model lifetimes of people, animals, and equipment.)

(a) Find the value of C, depending on both α and β, that makes $f(x)$ a probability density function.

(b) For the value of C found in part (a), find the value of the mean μ.

(c) For the value of C found in part (a), find the variance σ^2.

EXPL **56.** The **Laplace transform,** named after the French mathematician Pierre-Simon de Laplace (1749–1827), of a function $f(x)$ is given by $L\{f(t)\}(s) = \displaystyle\int_0^\infty f(t)e^{-st}\, dt$. Laplace transforms are useful for solving differential equations.

(a) Show that the Laplace transform of t^α is given by $\Gamma(\alpha + 1)/s^{\alpha+1}$ and is defined for $s > 0$.

(b) Show that the Laplace transform of $e^{\alpha t}$ is given by $1/(s - \alpha)$ and is defined for $s > \alpha$.

(c) Show that the Laplace transform of $\sin(\alpha t)$ is given by $\alpha/(s^2 + \alpha^2)$ and is defined for $s > 0$.

57. By interpreting each of the following integrals as an area and then calculating this area by a y-integration, evaluate:

(a) $\displaystyle\int_0^1 \sqrt{\frac{1 - x}{x}}\, dx$ (b) $\displaystyle\int_{-1}^1 \sqrt{\frac{1 + x}{1 - x}}\, dx$

EXPL **58.** Suppose that $0 < p < q$ and $\displaystyle\int_0^\infty \frac{1}{x^p + x^q}\, dx$ converges. What can you say about p and q?

Answers to Concepts Review: **1.** unbounded **2.** 2 **3.** $\displaystyle\lim_{b \to 4^-} \int_0^b \left(1/\sqrt{4 - x}\right) dx$ **4.** $p < 1$

CHAPTER 9

Infinite Series

9.1 Infinite Sequences
9.2 Infinite Series
9.3 Positive Series: The Integral Test
9.4 Positive Series: Other Tests
9.5 Alternating Series, Absolute Convergence, and Conditional Convergence
9.6 Power Series
9.7 Operations on Power Series
9.8 Taylor and Maclaurin Series
9.9 The Taylor Approximation to a Function

9.1 Infinite Sequences

In simple language, a sequence

$$a_1, a_2, a_3, a_4, \ldots$$

is an ordered arrangement of real numbers, one for each positive integer. More formally, an **infinite sequence** is a function whose domain is the set of positive integers and whose range is a set of real numbers. We may denote a sequence by $a_1, a_2, a_3, \ldots$, by $\{a_n\}_{n=1}^{\infty}$, or simply by $\{a_n\}$. Occasionally, we will extend the notion slightly by allowing the domain to consist of all integers greater than or equal to a specified integer, as in $b_0, b_1, b_2, \ldots$ and $c_8, c_9, c_{10}, \ldots$, which are also denoted by $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=8}^{\infty}$, respectively.

A sequence may be specified by giving enough initial terms to establish a pattern, as in

$$1, 4, 7, 10, 13, \ldots$$

by an **explicit formula** for the nth term, as in

$$a_n = 3n - 2, \qquad n \geq 1$$

or by a **recursion formula**

$$a_1 = 1, \qquad a_n = a_{n-1} + 3, \qquad n \geq 2$$

Note that each of our three illustrations describes the same sequence. Here are four more explicit formulas and the first few terms of the sequences that they generate.

(1) $a_n = 1 - \dfrac{1}{n}$, $\quad n \geq 1$: $\quad 0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$

(2) $b_n = 1 + (-1)^n \dfrac{1}{n}$, $\quad n \geq 1$: $\quad 0, \dfrac{3}{2}, \dfrac{2}{3}, \dfrac{5}{4}, \dfrac{4}{5}, \dfrac{7}{6}, \dfrac{6}{7}, \ldots$

(3) $c_n = (-1)^n + \dfrac{1}{n}$, $\quad n \geq 1$: $\quad 0, \dfrac{3}{2}, -\dfrac{2}{3}, \dfrac{5}{4}, -\dfrac{4}{5}, \dfrac{7}{6}, -\dfrac{6}{7}, \ldots$

(4) $d_n = 0.999$, $\quad n \geq 1$: $\quad 0.999, 0.999, 0.999, 0.999, \ldots$

Someone is sure to argue that there are many different sequences that begin

$$1, 4, 7, 10, 13$$

and we agree. For example, the formula

$$3n - 2 + (n-1) \cdot (n-2) \cdots (n-5)$$

generates those five numbers. Who but an expert would think of this formula? When we ask you to look for a pattern, we mean a simple and obvious pattern.

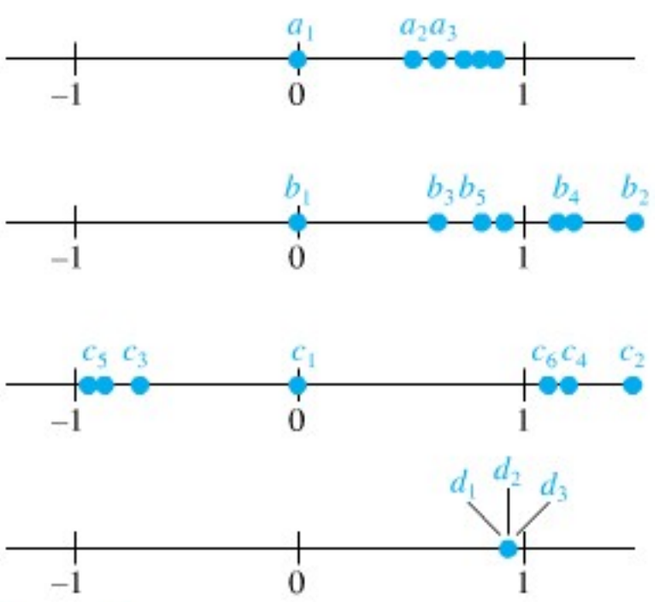


Figure 1

Convergence Consider the four sequences just defined. Each has values that pile up near 1 (see the diagrams in Figure 1). But do they all *converge* to 1? The correct response is that sequences $\{a_n\}$ and $\{b_n\}$ converge to 1, but $\{c_n\}$ and $\{d_n\}$ do not.

For a sequence to converge to 1 means first that values of the sequence should get close to 1. But they must do more than get close; they must *remain* close for all n beyond a certain value. This rules out sequence $\{c_n\}$. And close means arbitrarily close, that is, within *any* specified nonzero distance from 1, which rules out sequence $\{d_n\}$. While sequence $\{d_n\}$ does not converge to 1, it is correct to say that it converges to 0.999. Sequence $\{c_n\}$ does not converge at all; we say it diverges.

Here is the formal definition, which we first introduced in Section 1.5.

Definition

The sequence $\{a_n\}$ is said to **converge** to L, and we write

$$\lim_{n\to\infty} a_n = L$$

if for each positive number ε there is a corresponding positive number N such that

$$n \geq N \Rightarrow |a_n - L| < \varepsilon$$

A sequence that fails to converge to any finite number L is said to **diverge,** or to be divergent.

To see a relationship with limits at infinity (Section 1.5), consider graphing $a_n = 1 - 1/n$ and $a(x) = 1 - 1/x$. The only difference is that in the sequence case the domain is restricted to the positive integers. In the first case, we write $\lim_{n\to\infty} a_n = 1$; in the second, $\lim_{x\to\infty} a(x) = 1$. Note the interpretations of ε and N in the diagrams in Figure 2.

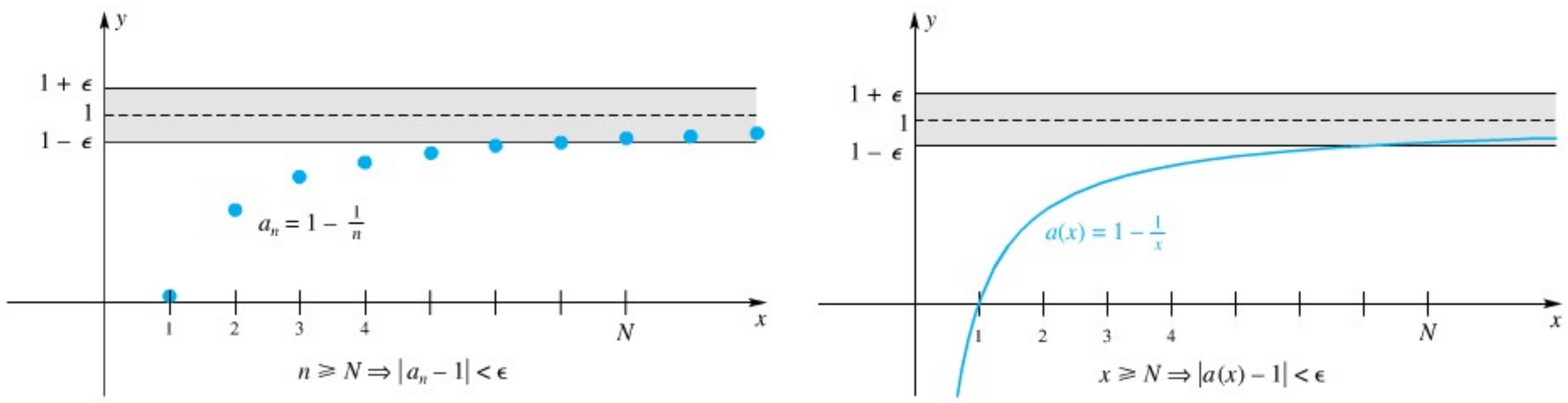


Figure 2

EXAMPLE 1 Show that if p is a positive integer, then

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

SOLUTION This is almost obvious from earlier work, but we can give a formal demonstration. Let an arbitrary $\varepsilon > 0$ be given. Choose N to be any number greater than $\sqrt[p]{1/\varepsilon}$. Then $n \geq N$ implies that

$$|a_n - L| = \left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\left(\sqrt[p]{1/\varepsilon}\right)^p} = \varepsilon$$

■

All the familiar limit theorems hold for convergent sequences. We state them without proof.

Theorem A Properties of Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and k a constant. Then

(i) $\lim_{n\to\infty} k = k$;

(ii) $\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n$;

(iii) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$;

(iv) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$;

(v) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided that $\lim_{n\to\infty} b_n \neq 0$.

EXAMPLE 2 Find $\lim_{n\to\infty} \dfrac{3n^2}{7n^2+1}$.

SOLUTION To decide what is happening to a quotient of two polynomials in n as n gets large, it is wise to divide the numerator and denominator by the largest power of n that occurs in the denominator. This justifies our first step below; the others are justified by appealing to statements from Theorem A as indicated by the circled numbers.

$$\lim_{n\to\infty} \frac{3n^2}{n^2+1} \overset{⑤}{=} \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} [7+(1/n^2)]} \overset{③}{=} \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} 7 + \lim_{n\to\infty} 1/n^2} \overset{①}{=} \frac{3}{7+\lim_{n\to\infty} 1/n^2} = \frac{3}{7+0} = \frac{3}{7}$$

By this time, the limit theorems are so familiar that we will normally jump directly from the first step to the final result. ■

EXAMPLE 3 Does the sequence $\{(\ln n)/e^n\}$ converge and, if so, to what number?

SOLUTION Here, and in many sequence problems, it is convenient to use the following almost obvious fact (see Figure 2).

> If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

This is convenient because we can apply l'Hôpital's Rule to the continuous variable problem. In particular, by l'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{1/x}{e^x} = 0$$

Thus,

$$\lim_{n\to\infty} \frac{\ln n}{e^n} = 0$$

That is, $\{(\ln n)/e^n\}$ converges to 0. ■

Here is another theorem that we have seen before in a slightly different guise (Theorem 1.3D).

≈ As Expected

In Chapter 8, we showed using L'Hôpital's Rule that e^x grows *faster* than any power of x and $\ln x$ grows *slower* than any power of x. Thus we would expect that $\ln x/e^x$ and $\ln n/e^n$ both go to 0.

Theorem B Squeeze Theorem

Suppose that $\{a_n\}$ and $\{c_n\}$ both converge to L and that $a_n \le b_n \le c_n$ for $n \ge K$ (K a fixed integer). Then $\{b_n\}$ also converges to L.

EXAMPLE 4 Show that $\lim_{n\to\infty} \dfrac{\sin^3 n}{n} = 0$.

SOLUTION For $n \ge 1$, $-1/n \le (\sin^3 n)/n \le 1/n$. Since $\lim_{n\to\infty}(-1/n) = 0$ and $\lim_{n\to\infty}(1/n) = 0$, the result follows by the Squeeze Theorem. ■

For sequences of variable sign, it is helpful to have the following result.

Theorem C

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof Since $-|a_n| \le a_n \le |a_n|$, the result follows from the Squeeze Theorem. ■

What happens to the numbers in the sequence $\{0.999^n\}$ as $n \to \infty$? We suggest that you calculate 0.999^n for $n = 10, 100, 1000$, and 10,000 on your calculator to make a good guess. Then note the following example.

EXAMPLE 5 Show that if $-1 < r < 1$ then $\lim_{n\to\infty} r^n = 0$.

SOLUTION If $r = 0$, the result is trivial, so suppose otherwise. Then $1/|r| > 1$, and so $1/|r| = 1 + p$ for some number $p > 0$. By the Binomial Formula,

$$\frac{1}{|r|^n} = (1 + p)^n = 1 + pn + (\text{positive terms}) \ge pn$$

Thus,

$$0 \le |r|^n \le \frac{1}{pn}$$

Since $\lim_{n\to\infty} (1/pn) = (1/p) \lim_{n\to\infty} (1/n) = 0$, it follows from the Squeeze Theorem that $\lim_{n\to\infty} |r|^n = 0$ or, equivalently, $\lim_{n\to\infty} |r^n| = 0$. By Theorem C, $\lim_{n\to\infty} r^n = 0$. ■

What if $r > 1$; for example, $r = 1.5$? Then r^n will march off toward ∞. In this case, we write

$$\lim_{n\to\infty} r^n = \infty, \qquad r > 1$$

However, we say that the sequence $\{r^n\}$ diverges. To converge, a sequence must approach a *finite* limit. The sequence $\{r^n\}$ also diverges when $r \le -1$.

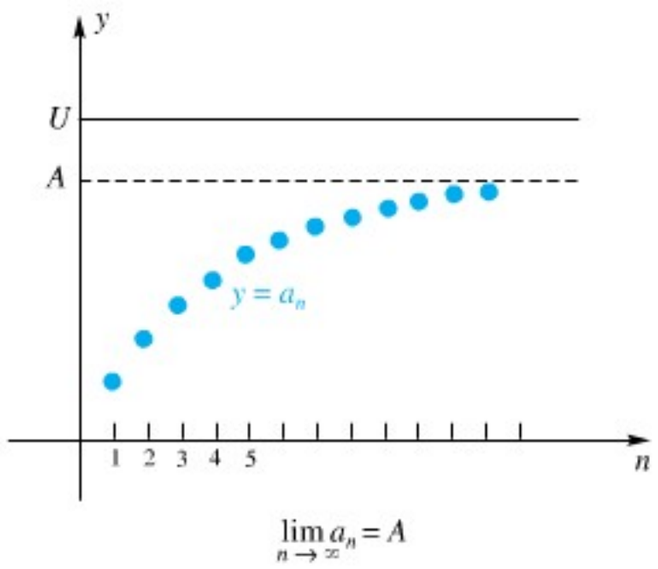


Figure 3

Monotonic Sequences Consider now an arbitrary **nondecreasing sequence** $\{a_n\}$, by which we mean $a_n \le a_{n+1}$, $n \ge 1$. One example is the sequence $a_n = n^2$; another is $a_n = 1 - 1/n$. If you think about it a little, you may convince yourself that such a sequence can do one of only two things. Either it marches off to infinity or, if it cannot do that because it is bounded above, then it must approach a lid (see Figure 3). Here is the formal statement of this very important result.

Theorem D Monotonic Sequence Theorem

If U is an upper bound for a nondecreasing sequence $\{a_n\}$, then the sequence converges to a limit A that is less than or equal to U. Similarly, if L is a lower bound for a nonincreasing sequence $\{b_n\}$, then the sequence $\{b_n\}$ converges to a limit B that is greater than or equal to L.

The expression **monotonic sequence** is used to describe either a nondecreasing or nonincreasing sequence; hence the name for this theorem.

Theorem D describes a very deep property of the real number system. It is equivalent to the *completeness property* of the real numbers, which in simple language says that the real line has no "holes" in it (see Problems 47 and 48). It is this property that distinguishes the real number line from the rational number line (which is full of holes). A great deal more could be said about this topic; we hope Theorem D appeals to your intuition and that you will accept it on faith until you take a more advanced course.

We make one more comment about Theorem D. It is not necessary that the sequences $\{a_n\}$ and $\{b_n\}$ be monotonic initially, only that they be monotonic from some point on, that is, for $n \geq K$. In fact, *the convergence or divergence of a sequence does not depend on the character of the initial terms, but rather on what is true for large n.*

EXAMPLE 6 Show that the sequence $b_n = n^2/2^n$ converges by using Theorem D.

SOLUTION The first few terms of this sequence are

$$\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \frac{9}{16}, \frac{49}{128}, \cdots$$

For $n \geq 3$ the sequence appears to be decreasing ($b_n > b_{n+1}$), a fact that we now establish. Each of the following inequalities is equivalent to the others.

$$\frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}}$$

$$n^2 > \frac{(n+1)^2}{2}$$

$$2n^2 > n^2 + 2n + 1$$

$$n^2 - 2n > 1$$

$$n(n-2) > 1$$

The last inequality is clearly true for $n \geq 3$. Since the sequence is decreasing (a stronger condition than nonincreasing) and is bounded below by zero, the Monotonic Sequence Theorem guarantees that it has a limit.

It would be easy using l'Hôpital's Rule to show that the limit is zero. ■

Concepts Review

1. An arrangement of numbers $a_1, a_2, a_3, \ldots$ is called _____.
2. We say the sequence $\{a_n\}$ converges if _____.
3. An increasing sequence that is also _____ must converge.
4. The sequence $\{r^n\}$ converges if and only if _____ $< r \leq$ _____.

Problem Set 9.1

In Problems 1–20, an explicit formula for a_n is given. Write the first five terms of $\{a_n\}$, determine whether the sequence converges or diverges, and, if it converges, find $\lim_{n\to\infty} a_n$.

1. $a_n = \dfrac{n}{3n-1}$

2. $a_n = \dfrac{3n+2}{n+1}$

3. $a_n = \dfrac{4n^2+2}{n^2+3n-1}$

4. $a_n = \dfrac{3n^2+2}{2n-1}$

5. $a_n = \dfrac{n^3+3n^2+3n}{(n+1)^3}$

6. $a_n = \dfrac{\sqrt{3n^2+2}}{2n+1}$

7. $a_n = (-1)^n \dfrac{n}{n+2}$

8. $a_n = \dfrac{n\cos(n\pi)}{2n-1}$

9. $a_n = \dfrac{\cos(n\pi)}{n}$

10. $a_n = e^{-n}\sin n$

11. $a_n = \dfrac{e^{2n}}{n^2+3n-1}$

12. $a_n = \dfrac{e^{2n}}{4^n}$

13. $a_n = \dfrac{(-\pi)^n}{5^n}$

14. $a_n = \left(\frac{1}{4}\right)^n + 3^{n/2}$

15. $a_n = 2 + (0.99)^n$

16. $a_n = \dfrac{n^{100}}{e^n}$

17. $a_n = \dfrac{\ln n}{\sqrt{n}}$

18. $a_n = \dfrac{\ln(1/n)}{\sqrt{2n}}$

19. $a_n = \left(1 + \dfrac{2}{n}\right)^{n/2}$

Hint: Theorem 6.5A.

20. $a_n = (2n)^{1/2n}$

In Problems 21–30, find an explicit formula $a_n =$ _____ for each sequence, determine whether the sequence converges or diverges, and, if it converges, find $\lim_{n\to\infty} a_n$.

21. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

22. $\dfrac{1}{2^2}, \dfrac{2}{2^3}, \dfrac{3}{2^4}, \dfrac{4}{2^5}, \cdots$

23. $-1, \frac{2}{3}, -\frac{3}{5}, \frac{4}{7}, -\frac{5}{9}, \ldots$

24. $1, \dfrac{1}{1-\frac{1}{2}}, \dfrac{1}{1-\frac{2}{3}}, \dfrac{1}{1-\frac{3}{4}}, \ldots$

25. $1, \dfrac{2}{2^2-1^2}, \dfrac{3}{3^2-2^2}, \dfrac{4}{4^2-3^2}, \ldots$

26. $\dfrac{1}{2-\frac{1}{2}}, \dfrac{2}{3-\frac{1}{3}}, \dfrac{3}{4-\frac{1}{4}}, \dfrac{4}{5-\frac{1}{5}}, \ldots$

27. $\sin 1, 2\sin\frac{1}{2}, 3\sin\frac{1}{3}, 4\sin\frac{1}{4}, \ldots$

28. $-\frac{1}{3}, \frac{4}{9}, -\frac{9}{27}, \frac{16}{81}, \ldots$

29. $2, 1, \dfrac{2^3}{3^2}, \dfrac{2^4}{4^2}, \dfrac{2^5}{5^2}, \ldots$

30. $1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \ldots$

In Problems 31–36, write the first four terms of the sequence $\{a_n\}$. Then use Theorem D to show that the sequence converges.

31. $a_n = \dfrac{4n-3}{2^n}$

32. $a_n = \dfrac{n}{n+1}\left(2-\dfrac{1}{n^2}\right)$

33. $a_n = \left(1-\dfrac{1}{4}\right)\left(1-\dfrac{1}{9}\right)\cdots\left(1-\dfrac{1}{n^2}\right), n \geq 2$

34. $a_n = 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$

35. $a_1 = 1, a_{n+1} = 1 + \frac{1}{2}a_n$

36. $a_1 = 2, a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$

C **37.** Assuming that $u_1 = \sqrt{3}$ and $u_{n+1} = \sqrt{3+u_n}$ determine a convergent sequence, find $\lim_{n\to\infty} u_n$ to four decimal places.

38. Show that $\{u_n\}$ of Problem 37 is bounded above and increasing. Conclude from Theorem D that $\{u_n\}$ converges. *Hint:* Use mathematical induction.

39. Find $\lim_{n\to\infty} u_n$ of Problem 37 algebraically. *Hint:* Let $u = \lim_{n\to\infty} u_n$. Then, since $u_{n+1} = \sqrt{3+u_n}$, $u = \sqrt{3+u}$. Now square both sides and solve for u.

40. Use the technique of Problem 39 to find $\lim_{n\to\infty} a_n$ of Problem 36.

C **41.** Assuming that $u_1 = 0$ and $u_{n+1} = 1.1^{u_n}$ determine a convergent sequence, find $\lim_{n\to\infty} u_n$ to four decimal places.

42. Show that $\{u_n\}$ of Problem 41 is increasing and bounded above by 2.

43. Find

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sin\frac{k}{n}\right)\frac{1}{n}$$

Hint: Write an equivalent definite integral.

44. Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{1+(k/n)^2}\right]\frac{1}{n} = \frac{\pi}{4}$$

45. Using the definition of limit, prove that $\lim_{n\to\infty} n/(n+1) = 1$; that is, for a given $\varepsilon > 0$, find N such that $n \geq N \Rightarrow |n/(n+1) - 1| < \varepsilon$.

46. As in Problem 45, prove that $\lim_{n\to\infty} n/(n^2+1) = 0$.

47. Let $S = \{x: x \text{ is rational and } x^2 < 2\}$. Convince yourself that S does not have a *least* upper bound in the rational numbers, but does have such a bound in the real numbers. In other words, the sequence of rational numbers $1, 1.4, 1.41, 1.414, \ldots$, has no limit within the rational numbers.

EXPL **48.** The *completeness property* of the real numbers says that for every set of real numbers that is bounded above, there exists a real number that is a least upper bound for the set. This property is usually taken as an axiom for the real numbers. Prove Theorem D using this property.

49. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded then $\lim_{n\to\infty} a_n b_n = 0$.

50. Prove that if $\{a_n\}$ converges and $\{b_n\}$ diverges then $\{a_n + b_n\}$ diverges.

51. If $\{a_n\}$ and $\{b_n\}$ both diverge, does it follow that $\{a_n + b_n\}$ diverges?

EXPL **52.** A famous sequence $\{f_n\}$, called the **Fibonacci Sequence** after Leonardo Fibonacci, who introduced it around A.D. 1200, is defined by the recursion formula

$$f_1 = f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n$$

(a) Find f_3 through f_{10}.

(b) Let $\phi = \frac{1}{2}(1+\sqrt{5}) \approx 1.618034$. The Greeks called this number the *golden ratio*, claiming that a rectangle whose dimensions were in this ratio was "perfect." It can be shown that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$
$$= \frac{1}{\sqrt{5}}[\phi^n - (-1)^n\phi^{-n}]$$

Check that this gives the right result for $n = 1$ and $n = 2$. The general result can be proved by induction (it is a nice challenge). More in line with this section, use this explicit formula to prove that $\lim_{n\to\infty} f_{n+1}/f_n = \phi$.

(c) Using the limit just proved, show that ϕ satisfies the equation $x^2 - x - 1 = 0$. Then, in another interesting twist, use the Quadratic Formula to show that the two roots of this equation are ϕ and $-1/\phi$, two numbers that occur in the explicit formula for f_n.

53. Consider an equilateral triangle containing $1 + 2 + 3 + \cdots + n = n(n+1)/2$ circles, each of diameter 1 and stacked as indicated in Figure 4 for the case $n = 4$. Find $\lim_{n\to\infty} A_n/B_n$, where A_n is the total area of the circles, and B_n is the area of the triangle.

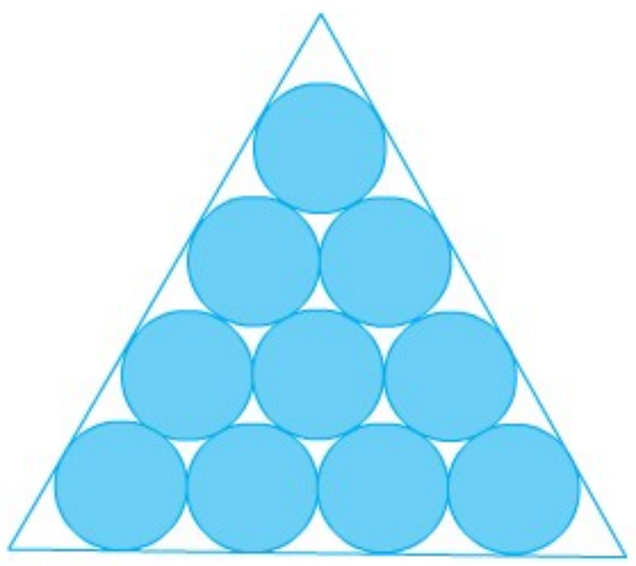


Figure 4

C *In Problems 54–59, use the fact that* $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\dfrac{1}{x}\right)$ *to find the limits.*

54. $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

55. $\lim_{n\to\infty}\left(1+\dfrac{1}{2n}\right)^n$

56. $\lim_{n\to\infty}\left(1+\dfrac{1}{n^2}\right)^n$

57. $\lim_{n\to\infty}\left(\dfrac{n-1}{n+1}\right)^n$

58. $\lim_{n\to\infty}\left(\dfrac{2+n^2}{3+n^2}\right)^n$

59. $\lim_{n\to\infty}\left(\dfrac{2+n^2}{3+n^2}\right)^{n^2}$

Answers to Concepts Review: **1.** a sequence **2.** $\lim_{n\to\infty} a_n$ exists (finite sense) **3.** bounded above **4.** $-1; 1$

9.2 Infinite Series

When you ask a computer or a calculator for the sine of an angle, or e to some power, it is using an algorithm to make this approximation. Many such algorithms are based on **infinite series,** the topic of this and subsequent sections in this chapter. We will see in Section 9.8 that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The topic of the last section of this chapter is the use of such series (actually, using just a finite number of terms in the series) to approximate a given function to a desired accuracy. But there is some work to do before reaching that point. We must give meaning to an "infinite" series like one of those above. We begin with the issue of an infinite series of numbers, as opposed to an infinite series of powers of x, called a power series (which we will address in Section 9.6). Consider for illustration the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

If we include just the first term, we have a "sum" of $\frac{1}{2}$. If we include the first two terms, we have a sum of $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$; if we include the first three terms, we have a sum of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$; and so on. This is the idea of a **partial sum,** that is the sum of a finite number of terms at the beginning of the series. We denote the nth partial sum, that is, the sum of the first n terms, by S_n. For this series the partial sums are

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Clearly, these partial sums get increasingly close to 1. In fact,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1$$

The infinite sum is then defined to be the limit of the partial sum S_n.

More generally, consider the **infinite series**

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

which is also denoted by $\sum_{k=1}^{\infty} a_k$ or $\sum a_k$. Then S_n, the ***n*th partial sum,** is given by

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

We make the following formal definition.

Definition

The infinite series $\sum_{k=1}^{\infty} a_k$ **converges** and has **sum** S if the sequence of partial sums $\{S_n\}$ converges to S. If $\{S_n\}$ diverges, then the series **diverges**. A divergent series has no sum.

Geometric Series A series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

where $a \neq 0$, is called a **geometric series.**

EXAMPLE 1 Show that a geometric series converges, and has sum $S = a/(1 - r)$ if $|r| < 1$, but diverges if $|r| \geq 1$.

SOLUTION Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. If $r = 1$, $S_n = na$, which grows without bound, and so $\{S_n\}$ diverges. If $r \neq 1$, we may write

$$S_n - rS_n = (a + ar + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n) = a - ar^n$$

and so

$$S_n = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} r^n$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ (Section 9.1, Example 5), and thus

$$S = \lim_{n\to\infty} S_n = \frac{a}{1 - r}$$

If $|r| > 1$ or $r = -1$, the sequence $\{r^n\}$ diverges, and consequently so does $\{S_n\}$. ■

EXAMPLE 2 Use the result of Example 1 to find the sum of the following two geometric series.

(a) $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots$

(b) $0.515151\ldots = \frac{51}{100} + \frac{51}{10{,}000} + \frac{51}{1{,}000{,}000} + \cdots$

SOLUTION

(a) $S = \dfrac{a}{1 - r} = \dfrac{\frac{4}{3}}{1 - \frac{1}{3}} = \dfrac{\frac{4}{3}}{\frac{2}{3}} = 2$

(b) $S = \dfrac{\frac{51}{100}}{1 - \frac{1}{100}} = \dfrac{\frac{51}{100}}{\frac{99}{100}} = \dfrac{51}{99} = \dfrac{17}{33}$

Incidently, the procedure in part (b) suggests how to show that any repeating decimal represents a rational number. ■

EXAMPLE 3 The diagram in Figure 1 represents an equilateral triangle containing infinitely many circles, tangent to the triangle and to neighboring circles, and reaching into the corners. What fraction of the area of the triangle is occupied by the circles?

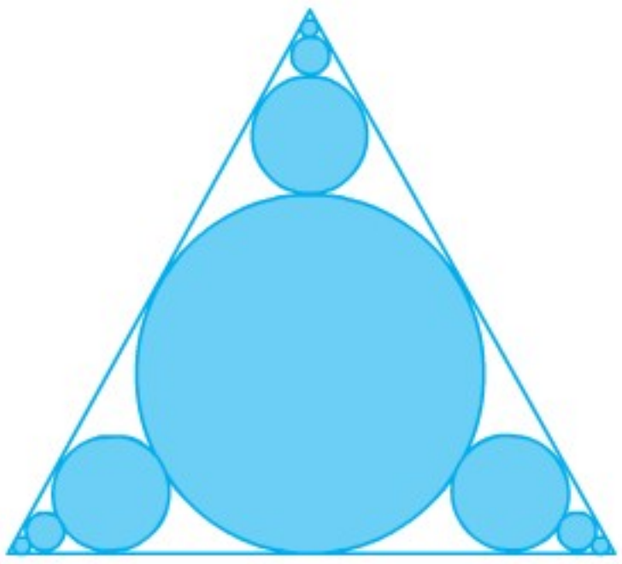


Figure 1

SOLUTION Suppose for convenience that the large circle has radius 1, which makes the triangle have sides of length $2\sqrt{3}$. Concentrate attention on the vertical stack of circles. With a bit of geometric reasoning (the center of the large circle is

two-thirds of the way from the upper vertex to the base), we see that the radii of these circles are $1, \frac{1}{3}, \frac{1}{9}, \ldots$ and conclude that the vertical stack has area

$$\pi\left[1^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{27}\right)^2 + \cdots\right]$$

$$= \pi\left[1 + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \cdots\right] = \pi\left[\frac{1}{1 - \frac{1}{9}}\right] = \frac{9\pi}{8}$$

The total area of all the circles is three times this number minus twice the area of the big circle, that is, $27\pi/8 - 2\pi$, or $11\pi/8$. Since the triangle has area $3\sqrt{3}$, the fraction of this area occupied by the circles is

$$\frac{11\pi}{24\sqrt{3}} \approx 0.83$$ ■

EXAMPLE 4 Suppose that Peter and Paul take turns tossing a fair coin until one of them tosses a head. If Peter goes first, what is the probability that he wins?

SOLUTION Peter can win by tossing a head on the first toss, which happens with probability $\frac{1}{2}$. Or he can win if these three events occur in succession: Peter tosses a tail, Paul tosses a tail, and Peter tosses a head. Each of these events has probability $\frac{1}{2}$, so they all occur with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. Another way Peter can win is for the first four tosses to be all tails, while Peter's third toss (the fifth overall) is a head. This occurs with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$. This process continues, so that the probability that Peter wins is the sum of the geometric series

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots = \frac{1/2}{1 - 1/4} = \frac{2}{3}$$

Paul therefore wins with probability $1 - \frac{2}{3} = \frac{1}{3}$. Peter has the greater chance of winning because he goes first. ■

Logic

Consider these two statements:

1. If $\sum_{k=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

The first statement is true for any sequence $\{a_n\}$; the second is not. This provides another example of a true statement (the first) whose *converse* is false.

Recall that the *contrapositive* of a statement is true whenever the statement is true. The contrapositive of the first statement is

3. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

A General Test for Divergence

Consider the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ once more. Its nth term a_n is given by $a_n = ar^{n-1}$. Example 1 shows that a geometric series converges *if and only if* $\lim_{n\to\infty} a_n = 0$.

Could this possibly be true of all series? The answer is no, although half of the statement (the "only-if" half) is correct. This leads to an important divergence test for series.

Theorem A ***n*th-Term Test for Divergence**

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist, then the series diverges.

Proof Let S_n be the nth partial sum and $S = \lim_{n\to\infty} S_n$. Note that $a_n = S_n - S_{n-1}$. Since $\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = S$, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$ ■

EXAMPLE 5 Show that $\sum_{n=1}^{\infty} \frac{n^3}{3n^3 + 2n^2}$ diverges.

SOLUTION

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{3n^3 + 2n^2} = \lim_{n\to\infty} \frac{1}{3 + 2/n} = \frac{1}{3}$$

Thus, by the nth-Term Test, the series diverges. ■

The Harmonic Series Students invariably want to turn Theorem A around and make it say that $a_n \to 0$ implies convergence of Σa_n. The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

shows that this is false. Clearly, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1/n) = 0$. However, the series diverges, as we now show.

EXAMPLE 6 Show that the harmonic series diverges.

SOLUTION We show that S_n grows without bound. Imagine n to be large and write

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \frac{1}{n} \\ &> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \cdots + \frac{1}{n} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n} \end{aligned}$$

It is clear that by taking n sufficiently large we can introduce as many $\frac{1}{2}$'s into the last expression as we wish. Thus, S_n grows without bound, and so $\{S_n\}$ diverges. Hence, the harmonic series diverges. ■

Collapsing Series A geometric series is one of the few series where we can actually give an explicit formula for S_n; a **collapsing series** is another (see Example 2 of Section 4.1).

EXAMPLE 7 Show that the following series converges and find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$$

SOLUTION Use a partial fraction decomposition to write

$$\frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3}$$

Then

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) \\ &= \frac{1}{3} - \frac{1}{n+3} \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} S_n = \tfrac{1}{3}$$

The series converges and has sum $\frac{1}{3}$. ■

A Note on Terminology

This theorem introduces a subtle shift in terminology. The symbol $\sum_{k=1}^{\infty} a_k$ is now being used both for the infinite series $a_1 + a_2 + \cdots$ and for the sum of this series, which is a number.

Properties of Convergent Series Convergent series behave much like finite sums; what you expect to be true often *is* true.

Theorem B Linearity of Convergent Series

If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, and if c is a constant, then $\sum_{k=1}^{\infty} ca_k$ and $\sum_{k=1}^{\infty} (a_k + b_k)$ also converge, and

(i) $\displaystyle\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k;$

(ii) $\displaystyle\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$

Proof By hypothesis, $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ and $\lim_{n\to\infty} \sum_{k=1}^{n} b_k$ both exist. Thus, use the properties of sums with finitely many terms and the properties of limits of sequences.

$$\text{(i)} \quad \sum_{k=1}^{\infty} ca_k = \lim_{n\to\infty} \sum_{k=1}^{n} ca_k = \lim_{n\to\infty} c\sum_{k=1}^{n} a_k$$

$$= c \lim_{n\to\infty} \sum_{k=1}^{n} a_k = c\sum_{k=1}^{\infty} a_k$$

$$\text{(ii)} \quad \sum_{k=1}^{\infty} (a_k + b_k) = \lim_{n\to\infty} \sum_{k=1}^{n} (a_k + b_k) = \lim_{n\to\infty} \left[\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \right]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \lim_{n\to\infty} \sum_{k=1}^{n} b_k = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \qquad \blacksquare$$

EXAMPLE 8 Calculate $\displaystyle\sum_{k=1}^{\infty} \left[3\left(\tfrac{1}{8}\right)^k - 5\left(\tfrac{1}{3}\right)^k\right]$.

SOLUTION By Theorem B and Example 1,

$$\sum_{k=1}^{\infty} \left[3\left(\frac{1}{8}\right)^k - 5\left(\frac{1}{3}\right)^k\right] = 3\sum_{k=1}^{\infty} \left(\frac{1}{8}\right)^k - 5\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$$

$$= 3\,\frac{\frac{1}{8}}{1 - \frac{1}{8}} - 5\,\frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{3}{7} - \frac{5}{2} = -\frac{29}{14} \qquad \blacksquare$$

Theorem C

If $\sum_{k=1}^{\infty} a_k$ diverges and $c \neq 0$, then $\sum_{k=1}^{\infty} ca_k$ diverges.

We leave the proof of this theorem to you (Problem 41). It implies, for example, that

$$\sum_{k=1}^{\infty} \frac{1}{3k} = \sum_{k=1}^{\infty} \frac{1}{3} \cdot \frac{1}{k}$$

diverges, since we know that the harmonic series diverges.

The associative law of addition allows us to group terms in a *finite* sum in any way that we please. For example,

$$2 + 7 + 3 + 4 + 5 = (2 + 7) + (3 + 4) + 5 = 2 + (7 + 3) + (4 + 5)$$

But sometimes we lose sight of the definition of an *infinite* series as the limit of a sequence of partial sums, and we let our intuition guide us into a paradox. For example, the series

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots$$

has partial sums

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 - 1 = 0 \\ S_3 &= 1 - 1 + 1 = 1 \\ S_4 &= 1 - 1 + 1 - 1 = 0 \\ &\vdots \end{aligned}$$

The sequence of partial sums, $1, 0, 1, 0, 1, \ldots$, diverges; thus the series $1 - 1 + 1 - 1 + \cdots$ diverges. We might, however, view the series as

$$(1 - 1) + (1 - 1) + \cdots$$

and claim that the sum is 0. Alternatively, we might view the series as

$$1 - (1 - 1) - (1 - 1) - \cdots$$

and claim that the sum is 1. The sum of the series cannot be equal to both 0 and 1. It turns out that grouping of terms in a series is acceptable provided that the series is convergent; in such a case we can group terms in any way that we wish.

Theorem D **Grouping Terms in an Infinite Series**

The terms of a convergent series can be grouped in any way (provided that the order of the terms is maintained), and the new series will converge with the same sum as the original series.

Proof Let Σa_n be the original convergent series and let $\{S_n\}$ be its sequence of partial sums. If Σb_m is a series formed by grouping the terms of Σa_n and if $\{T_m\}$ is its sequence of partial sums, then each T_m is one of the S_n's. For example, T_4 might be

$$T_4 = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6) + (a_7 + a_8)$$

in which case $T_4 = S_8$. Thus, $\{T_m\}$ is a "subsequence" of $\{S_n\}$. A moment's thought should convince you that if $S_n \to S$ then $T_m \to S$. ■

Concepts Review

1. An expression of the form $a_1 + a_2 + a_3 + \cdots$ is called _____.

2. A series $a_1 + a_2 + \cdots$ is said to converge if the sequence $\{S_n\}$ converges, where $S_n =$ _____.

3. The geometric series $a + ar + ar^2 + \cdots$ converges if _____; in this case the sum of the series is _____.

4. If $\lim_{n\to\infty} a_n \neq 0$, we can be sure that the series $\sum_{n=1}^{\infty} a_n$ _____.

Problem Set 9.2

In Problems 1–14, indicate whether the given series converges or diverges. If it converges, find its sum. Hint: It may help you to write out the first few terms of the series.

1. $\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)^k$

2. $\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^{-k-2}$

3. $\sum_{k=0}^{\infty}\left[2\left(\frac{1}{4}\right)^k + 3\left(-\frac{1}{5}\right)^k\right]$

4. $\sum_{k=1}^{\infty}\left[5\left(\frac{1}{2}\right)^k - 3\left(\frac{1}{7}\right)^{k+1}\right]$

5. $\sum_{k=1}^{\infty}\frac{k-5}{k+2}$

6. $\sum_{k=1}^{\infty}\left(\frac{9}{8}\right)^k$

7. $\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right)$

8. $\sum_{k=1}^{\infty}\frac{3}{k}$

9. $\sum_{k=1}^{\infty}\frac{k!}{100^k}$

10. $\sum_{k=1}^{\infty}\frac{2}{(k+2)k}$

11. $\sum_{k=1}^{\infty}\left(\frac{e}{\pi}\right)^{k+1}$

12. $\sum_{k=1}^{\infty}\frac{4^{k+1}}{7^{k-1}}$

13. $\sum_{k=2}^{\infty}\left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right)$

14. $\sum_{k=6}^{\infty}\frac{2}{k-5}$

In Problems 15–20, write the given decimal as an infinite series, then find the sum of the series, and finally, use the result to write the decimal as a ratio of two integers (see Example 2).

15. 0.22222...

16. 0.21212121...

17. 0.013013013...

18. 0.125125125...

19. 0.49999...

20. 0.36717171...

21. Evaluate $\sum_{k=0}^{\infty} r(1-r)^k, 0 < r < 2.$

22. Evaluate $\sum_{k=0}^{\infty}(-1)^k x^k, -1 < x < 1.$

23. Show that $\sum_{k=1}^{\infty}\ln\frac{k}{k+1}$ diverges. *Hint:* Obtain a formula for S_n.

24. Show that $\sum_{k=2}^{\infty}\ln\left(1-\frac{1}{k^2}\right) = -\ln 2.$

25. A ball is dropped from a height of 100 feet. Each time it hits the floor, it rebounds to $\frac{2}{3}$ its previous height. Find the total distance it travels before coming to rest.

26. Three people, A, B, and C, divide an apple as follows. First they divide it into fourths, each taking a quarter. Then they divide the leftover quarter into fourths, each taking a quarter, and so on. Show that each gets a third of the apple.

27. Suppose that the government pumps an extra $1 billion into the economy. Assume that each business and individual saves 25% of its income and spends the rest, so of the initial $1 billion, 75% is respent by individuals and businesses. Of that amount, 75% is spent, and so forth. What is the total increase in spending due to the government action? (This is called the *multiplier effect* in economics.)

28. Do Problem 27 assuming that only 10% of the income is saved at each stage.

≈ **29.** Assume that square $ABCD$ (Figure 2) has sides of length 1 and that E, F, G, and H are the midpoints of the sides. If the indicated pattern is continued indefinitely, what will be the area of the painted region?

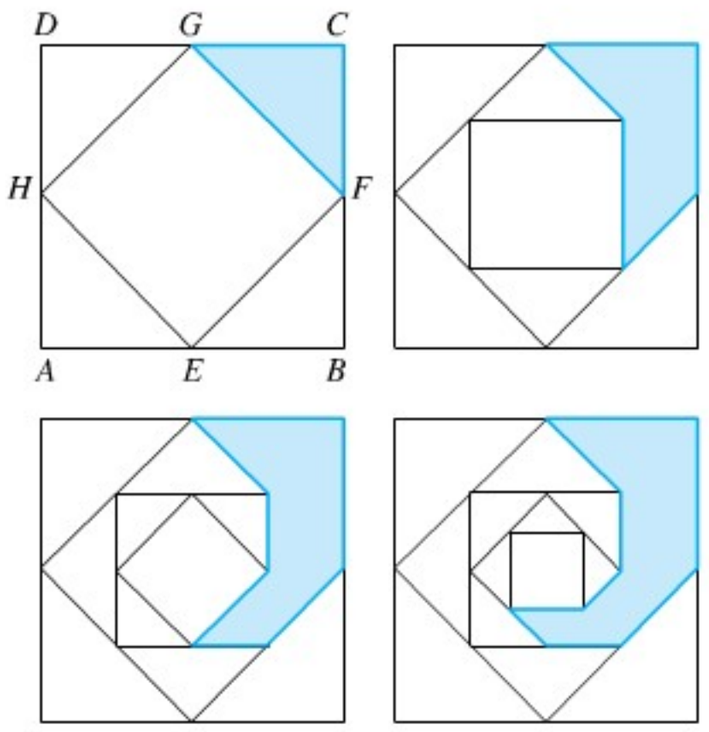


Figure 2

≈ **30.** If the pattern shown in Figure 3 is continued indefinitely, what fraction of the original square will eventually be painted?

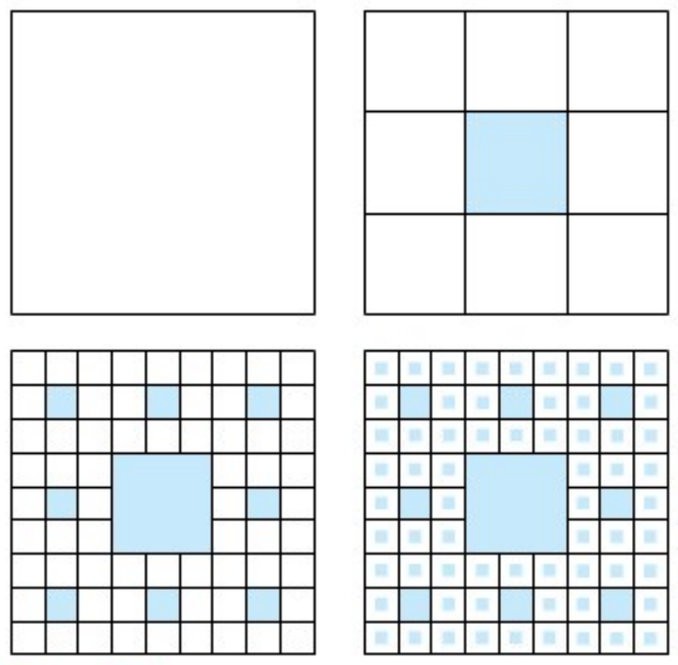


Figure 3

≈ **31.** Each triangle in the descending chain (Figure 4) has its vertices at the midpoints of the sides of the next larger one. If the indicated pattern of painting is continued indefinitely, what fraction of the original triangle will be painted? Does the original triangle need to be equilateral for this to be true?

Figure 4

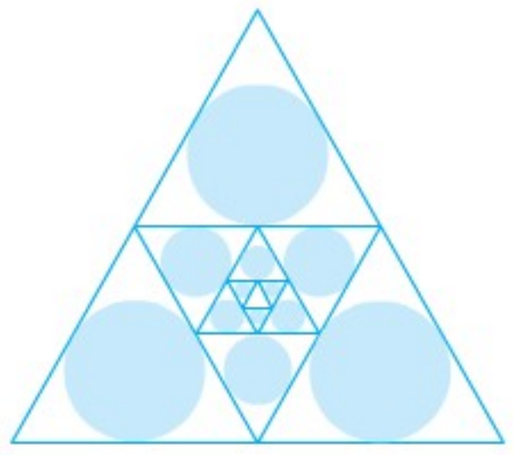


Figure 5

≈ **32.** Circles are inscribed in the triangles of Problem 31 as indicated in Figure 5. If the original triangle is equilateral, what fraction of the area is eventually painted?

33. The Koch snowflake is formed as follows. Begin with an equilateral triangle, which we'll assume has sides of length 9. On each side, replace the middle third with two sides of an equilateral triangle having sides of length 3. Then on each of these 12 sides replace the middle third with two sides of an equilateral triangle having sides of length 1. The Koch snowflake is the result of continuing this process indefinitely. The first four stages are shown in Figure 6.

(a) Find the perimeter of the Koch snowflake or show that it is infinite.

(b) Find the area of the Koch snowflake or show that it is infinite.

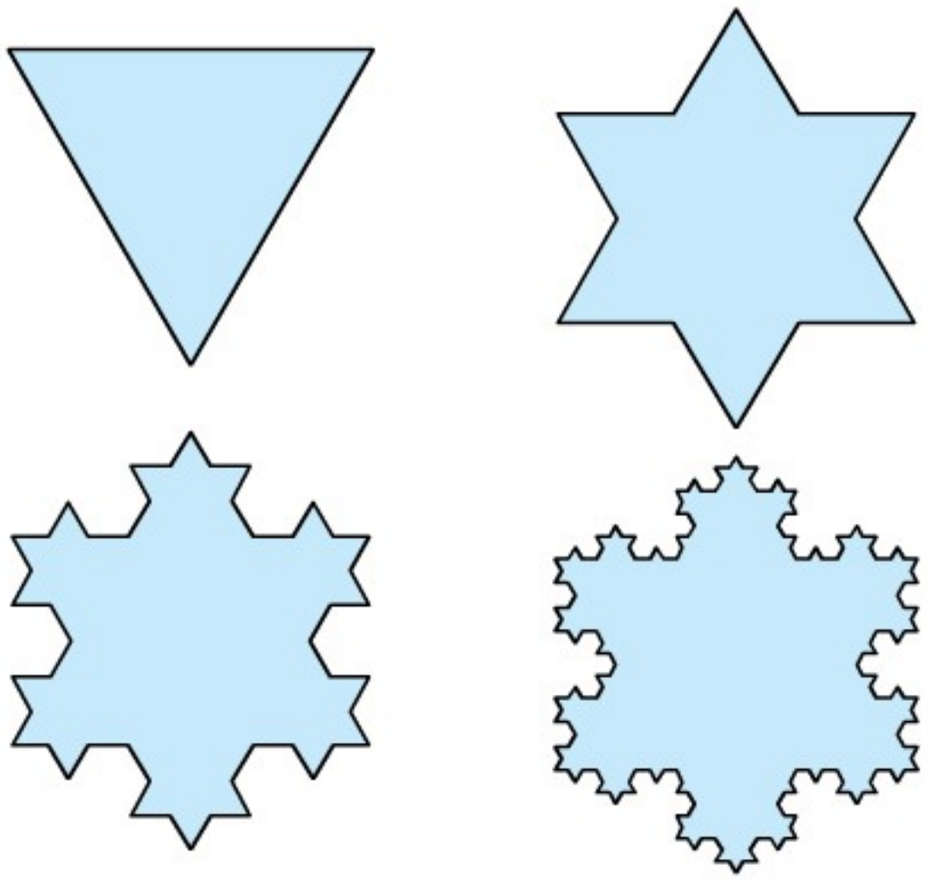


Figure 6

34. Consider the right triangle ABC as shown in Figure 7. Point A_1 is determined by drawing a perpendicular to line AB through C; B_1 is formed by drawing a line parallel to AC through A_1. This process is continued to produce $A_2, A_3, \ldots$, and $B_2, B_3, \ldots$. Find a series for the areas of the triangles formed in this way and show that the series sums to the area of ΔABC.

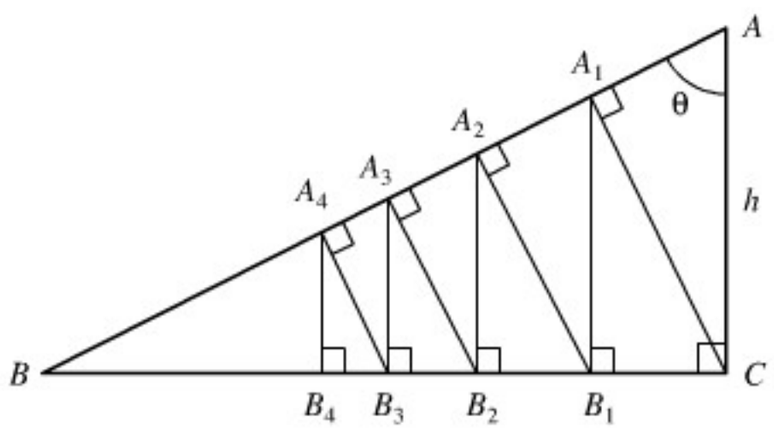


Figure 7

35. In one version of Zeno's paradox, Achilles can run ten times as fast as the tortoise, but the tortoise has a 100-yard head-start. Achilles cannot catch the tortoise, says Zeno, because when Achilles runs 100 yards the tortoise will have moved 10 yards ahead, when Achilles runs another 10 yards, the tortoise will have moved 1 yard ahead, and so on. Convince Zeno that Achilles will catch the tortoise and tell him exactly how many yards Achilles will have to run to do it.

36. Tom and Joel are good runners, both able to run at a constant speed of 10 miles per hour. Their amazing dog Trot can do even better; he runs at 20 miles per hour. Starting from towns 60 miles apart, Tom and Joel run toward each other while Trot runs back and forth between them. How far does Trot run by the time the boys meet? Assume that Trot started with Tom running toward Joel and that he is able to make instant turnarounds. Solve the problem two ways.

(a) Use a geometric series.

(b) Find a shorter way to do the problem.

37. Suppose that Peter and Paul alternate tossing a coin for which the probability of a head is $\frac{1}{3}$ and the probability of a tail is $\frac{2}{3}$. If they toss until someone gets a head, and Peter goes first, what is the probability that Peter wins?

38. Repeat Problem 37 for the case where the probability of a head is p and the probability of a tail is $1 - p$.

39. Suppose that Mary rolls a fair die until a "6" occurs. Let X denote the random variable that is the number of tosses needed for this "6" to occur. Find the probability distribution for X and verify that all the probabilities sum to 1.

40. Use the fact that

$$\sum_{x=1}^{\infty} xp^x = \frac{p}{(1-p)^2}$$

(which we will derive in Section 9.7) to find the expected value of the random variable X in Problem 39.

41. Prove: If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} ca_k$ for $c \neq 0$.

42. Use Problem 41 to conclude that $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$ diverges.

43. Suppose that one has an unlimited supply of identical blocks each 1 unit long.

(a) Show that they may be stacked as in Figure 8 without toppling. *Hint:* Consider centers of mass.

(b) How far can one make the top block protrude to the right of the bottom block using this method of stacking?

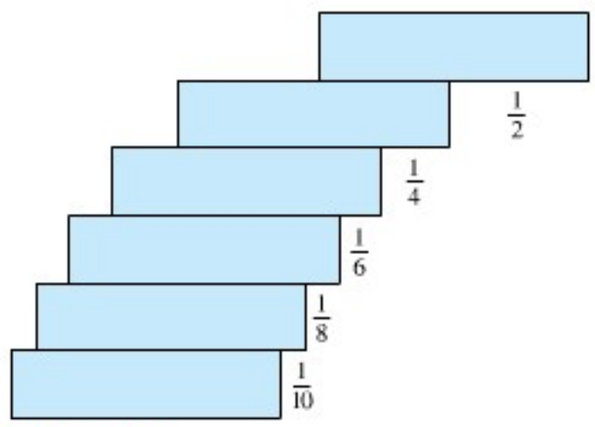


Figure 8

44. How large must N be in order for $S_N = \sum_{k=1}^{N} (1/k)$ just to exceed 4? *Note:* Computer calculations show that for S_N to exceed 20, $N = 272{,}400{,}600$, and for S_N to exceed 100, $N \approx 1.5 \times 10^{43}$.

45. Prove that if Σa_n diverges and Σb_n converges, then $\Sigma(a_n + b_n)$ diverges.

46. Show that it is possible for Σa_n and Σb_n both to diverge and yet for $\Sigma(a_n + b_n)$ to converge.

47. By looking at the region in Figure 9 first vertically and then horizontally, conclude that

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{3}{8} + \tfrac{4}{16} + \cdots$$

and use this fact to calculate:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$

(b) $\bar{x}$, the horizontal coordinate of the centroid of the region.

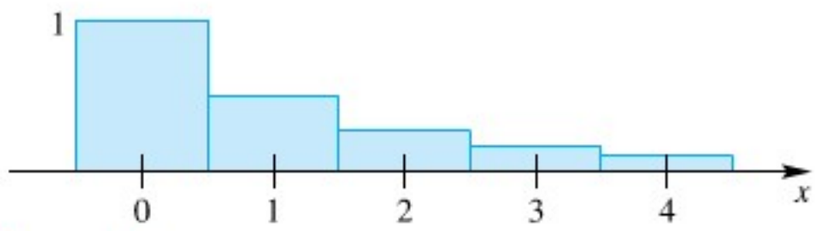


Figure 9

48. Let r be a fixed number with $|r| < 1$. Then it can be shown that $\displaystyle\sum_{k=1}^{\infty} kr^k$ converges, say with sum S. Use the properties of Σ to show that

$$(1 - r)S = \sum_{k=1}^{\infty} r^k$$

and then obtain a formula for S, thus generalizing Problem 47a.

49. Many drugs are eliminated from the body in an exponential manner. Thus, if a drug is given in dosages of size C at time intervals of length t, the amount A_n of the drug in the body just after the $(n + 1)$st dose is

$$A_n = C + Ce^{-kt} + Ce^{-2kt} + \cdots + Ce^{-nkt}$$

where k is a positive constant that depends on the type of drug.

(a) Derive a formula for A, the amount of drug in the body just after a dose, if a person has been on the drug for a very long time (assume an infinitely long time).

(b) Evaluate A if it is known that one-half of a dose is eliminated from the body in 6 hours and doses of size 2 milligrams are given every 12 hours.

50. Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{2^k}{(2^{k+1} - 1)(2^k - 1)}$$

51. Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{1}{f_k f_{k+2}}$ where $\{f_k\}$ is the Fibonacci sequence introduced in Problem 52 of Section 9.1. *Hint:* First show that

$$\frac{1}{f_k f_{k+2}} = \frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}}$$

Answers to Concepts Review: **1.** an infinite series **2.** $a_1 + a_2 + a_3 + \cdots + a_n$ **3.** $|r| < 1$; $a/(1 - r)$ **4.** diverges

9.3 Positive Series: The Integral Test

We introduced some important ideas in Section 9.2, but we illustrated them mainly for two very special types of series: geometric series and collapsing series. For these series we can give exact formulas for the partial sums S_n, something that we can rarely do for most other types of series. Our task now is to begin a study of very general infinite series.

There are always two important questions to ask about a series.

1. Does the series converge?
2. If it converges, what is its sum?

How shall we answer these questions? Someone may suggest that we use a computer. To answer the first question, simply add up more and more terms of the series, watching the numbers you get as partial sums. If these numbers seem to settle down on a fixed number S, the series converges. And in this case, S is the sum of the series, answering the second question. This response is plain wrong for question 1 and only partially adequate for question 2. Let us see why.

Consider the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

introduced in Section 9.2 and discussed in Example 6 and Problem 44 of that section. We know that this series diverges, but a computer would not help us to discover this fact. The partial sums S_n of this series grow without bound, but they grow so slowly that it takes over 272 million terms for S_n to reach 20 and over 10^{43} terms for S_n to reach 100. Because of the inherent limitation in the number of digits that it can handle, a computer would eventually give repeated values for S_n, suggesting wrongly that the S_n's were converging. What is true for the harmonic series is true for any slowly diverging series. We state it emphatically: A computer is no substitute for mathematical tests of convergence and divergence, a subject to which we now turn.

Important Reminders

$$a_1, a_2, a_3, \ldots$$

is a *sequence*.

$$a_1 + a_2 + a_3 + \cdots$$

is a *series*.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is the nth *partial sum* of the series.

$$S_1, S_2, S_3, \ldots$$

is the *sequence of partial sums* of the series. The series *converges* if and only if

$$S = \lim_{n \to \infty} S_n$$

exists and is finite, in which case S is called the *sum* of the series.

In this and the next section, we restrict our attention to series with positive (or at least nonnegative) terms. With this restriction, we will be able to give some remarkably simple convergence tests. Tests for series with terms of arbitrary sign are presented in Section 9.5.

Bounded Partial Sums Our first result flows directly from the Monotonic Sequence Theorem (Theorem 9.1D).

Theorem A Bounded Sum Test

A series Σa_k of nonnegative terms converges if and only if its partial sums are bounded above.

Proof As usual, let $S_n = a_1 + a_2 + \cdots + a_n$. Since $a_k \geq 0$, $S_{n+1} \geq S_n$; that is, $\{S_n\}$ is a nondecreasing sequence. Thus, by Theorem 9.1D, the sequence $\{S_n\}$ will converge provided that there is a number U such that $S_n \leq U$ for all n. Otherwise, the S_n's will grow without bound, in which case $\{S_n\}$ diverges. ■

EXAMPLE 1 Show that the series $\dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots$ converges.

SOLUTION We aim to show that the partial sums S_n are bounded above. Note first that

$$n! = 1 \cdot 2 \cdot 3 \cdots n \geq 1 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1}$$

and so $1/n! \leq 1/2^{n-1}$. Thus,

$$\begin{aligned} S_n &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \end{aligned}$$

These latter terms come from a geometric series with $r = \frac{1}{2}$. They can be added by a formula in the solution of Example 1 of Section 9.2. We obtain

$$S_n \leq \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2\left[1 - \left(\frac{1}{2}\right)^n\right] < 2$$

Thus, by the Bounded Sum Test, the given series converges. The argument also shows that its sum S is at most 2. Later we will show that $S = e - 1 \approx 1.71828$. ■

Series and Improper Integrals The behavior of $\displaystyle\sum_{k=1}^{\infty} f(k)$ and $\displaystyle\int_1^{\infty} f(x)\,dx$ with respect to convergence is similar and gives a very powerful test.

Theorem B Integral Test

Let f be a continuous, positive, nonincreasing function on the interval $[1, \infty)$ and suppose that $a_k = f(k)$ for all positive integers k. Then the infinite series

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx$$

converges.

We remark that the integer 1 may be replaced by any positive integer M throughout this theorem (see Example 4).

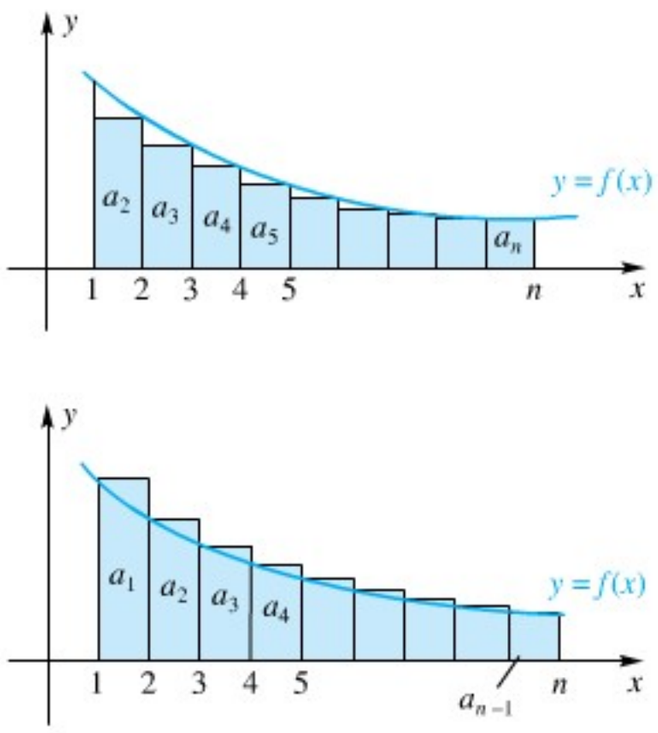


Figure 1

Proof The diagrams in Figure 1 indicate how we may interpret the partial sums of the series Σa_k as areas and thereby relate the series to a corresponding integral. Note that the area of each rectangle is equal to its height, since the width is 1 in each case. From these diagrams, we easily see that

$$\sum_{k=2}^{n} a_k \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} a_k$$

Now suppose that $\displaystyle\int_1^\infty f(x)\,dx$ converges. Then, by the left inequality above,

$$S_n = a_1 + \sum_{k=2}^{n} a_k \leq a_1 + \int_1^n f(x)\,dx \leq a_1 + \int_1^\infty f(x)\,dx$$

Therefore, by the Bounded Sum Test, $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

On the other hand, suppose that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. Then, by the right inequality above, if $t \leq n$,

$$\int_1^t f(x)\,dx \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} a_k \leq \sum_{k=1}^{\infty} a_k$$

Since $\displaystyle\int_1^t f(x)\,dx$ increases with t and is bounded above, $\displaystyle\lim_{t\to\infty}\int_1^t f(x)\,dx$ must exist; that is, $\displaystyle\int_1^\infty f(x)\,dx$ converges. ■

The conclusion to Theorem B is often stated this way. *The series* $\displaystyle\sum_{k=1}^{\infty} f(k)$ *and the improper integral* $\displaystyle\int_1^\infty f(x)\,dx$ *converge or diverge together.* You should see that this is equivalent to our statement.

EXAMPLE 2 **(*p*-Series Test)** The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

where p is a constant, is called a ***p*-series.** Show each of the following:

(a) The p-series converges if $p > 1$.
(b) The p-series diverges if $p \leq 1$.

SOLUTION If $p \geq 0$, the function $f(x) = 1/x^p$ is continuous, positive, and nonincreasing on $[1, \infty)$ and $f(k) = 1/k^p$. Thus, by the Integral Test, $\Sigma(1/k^p)$ converges if and only if $\displaystyle\lim_{t\to\infty}\int_1^t x^{-p}\,dx$ exists (as a finite number).

If $p \neq 1$,

$$\int_1^t x^{-p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{t^{1-p} - 1}{1-p}$$

If $p = 1$,

$$\int_1^t x^{-1}\,dx = [\ln x]_1^t = \ln t$$

Since $\displaystyle\lim_{t\to\infty} t^{1-p} = 0$ if $p > 1$ and $\displaystyle\lim_{t\to\infty} t^{1-p} = \infty$ if $p < 1$, and since $\displaystyle\lim_{t\to\infty} \ln t = \infty$, we conclude that the p-series converges if $p > 1$ and diverges if $0 \leq p \leq 1$.

The Tail of a Series

The beginning of a series plays no role in its convergence or divergence. Only the tail is important (the tail really does wag the dog). By the *tail* of a series, we mean

$$a_N + a_{N+1} + a_{N+2} + \cdots$$

where N denotes an arbitrarily large number. Hence, in testing for convergence or divergence of a series, we can ignore the beginning terms or even change them. Clearly, however, the sum of a series does depend on all its terms, including the initial ones.

We still have the case $p < 0$ to consider. In this case, the nth term of $\Sigma(1/k^p)$, that is, $1/n^p$, does not even tend toward 0. Thus, by the nth-Term Test, the series diverges.

Note that the case $p = 1$ gives the harmonic series, which was treated in Section 9.2. Our results here and there are consistent. The harmonic series diverges. ■

EXAMPLE 3 Does $\displaystyle\sum_{k=4}^{\infty} \frac{1}{k^{1.001}}$ converge or diverge?

SOLUTION By the p-Series Test, $\displaystyle\sum_{k=1}^{\infty}(1/k^{1.001})$ converges. *The insertion or removal of a finite number of terms in a series cannot affect its convergence or divergence (though it may affect the sum).* Thus, the given series converges. ■

EXAMPLE 4 Determine whether $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ converges or diverges.

SOLUTION The hypotheses of the Integral Test are satisfied for $f(x) = 1/(x \ln x)$ on $[2, \infty)$. That the interval is $[2, \infty)$ rather than $[1, \infty)$ is inconsequential, as we noted right after Theorem B. Now,

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{\ln x}\left(\frac{1}{x}\, dx\right) = \lim_{t\to\infty} [\ln \ln x]_2^t = \infty$$

Thus, $\Sigma 1/(k \ln k)$ diverges. ■

Approximating the Sum of a Series So far we have been concerned with whether a series converges or diverges. Except for a few special cases, such as the geometric series, or a collapsing series, we have not addressed the question of what a series converges to, if it converges. This is, in general, a difficult question, but at this point we can use the method suggested by the integral test to *approximate* the sum of a series.

If we use the nth partial sum S_n to approximate the sum of the series

$$S = a_1 + a_2 + a_3 + \cdots$$

then the error we make is

$$E_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$$

Let $f(x)$ be a function with the properties that $a_n = f(n)$, and f is positive, continuous, and nonincreasing on $[1, \infty)$; these are the conditions of Theorem B. Under these conditions

$$E_n = a_{n+1} + a_{n+2} + \cdots < \int_n^{\infty} f(x)\, dx$$

(see Figure 2). We can use this result to find an upper bound on the error involved in using the first n terms to approximate the sum S of the series, and we can use it to determine how large n must be to approximate S to a desired accuracy.

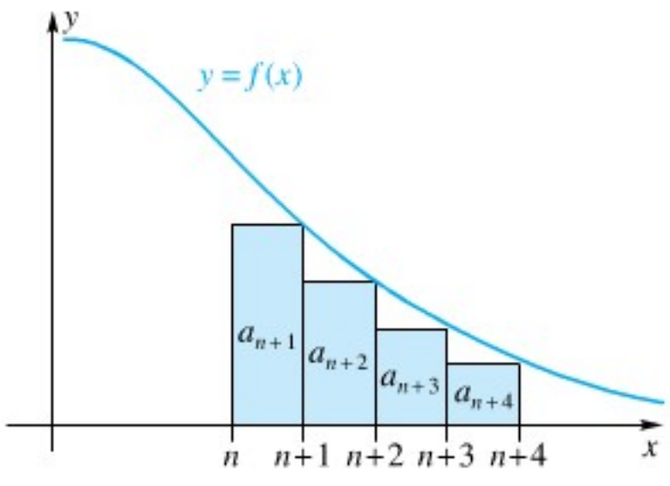


Figure 2

EXAMPLE 5 Find an upper bound for the error in using the sum of the first twenty terms to approximate the sum of the convergent series $S = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$.

SOLUTION The obvious choice for $f(x)$ is $f(x) = 1/x^{3/2}$; this function is positive, continuous, and nonincreasing on $[1, \infty)$. The error satisfies

$$E_{20} = \sum_{k=20+1}^{\infty} \frac{1}{k^{3/2}} < \int_{20}^{\infty} \frac{1}{x^{3/2}}\, dx = \lim_{A\to\infty} \left[-2x^{-1/2}\right]_{20}^{A} = \frac{2}{\sqrt{20}} \approx 0.44721$$

Even with twenty terms, the error is somewhat large. ■

EXAMPLE 6 How large must n be so that the partial sum S_n approximates the sum of the series in Example 5 with an error of no more than 0.005?

SOLUTION The error satisfies

$$E_n = \sum_{k=n+1}^{\infty} \frac{1}{k^{3/2}} < \int_n^{\infty} \frac{1}{x^{3/2}}\,dx = \lim_{A\to\infty}\left[-2x^{-1/2}\right]_n^A = \frac{2}{\sqrt{n}}$$

Thus, in order to guarantee that the error is less than 0.005, we need to have

$$\frac{2}{\sqrt{n}} < 0.005$$

$$\sqrt{n} > \frac{2}{0.005}$$

$$n > \left(\frac{2}{0.005}\right)^2 = 400^2 = 160{,}000$$

■

Concepts Review

1. A series of nonnegative terms converges if and only if its partial sums are _____.

2. The Integral Test relates the convergence of $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$, assuming $a_k =$ _____ and f is _____, _____, and _____ on $[1, \infty)$.

3. The insertion or removal of a finite number of terms in a series does not affect its _____, although it may affect its sum.

4. The p-series $\sum_{k=1}^{\infty} (1/k^p)$ converges if and only if _____.

Problem Set 9.3

Use the Integral Test to determine the convergence or divergence of each of the following series.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k+3}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2k-3}$

3. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^2+3}$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2k^2+1}$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{-2}{\sqrt{k+2}}$

6. $\displaystyle\sum_{k=100}^{\infty} \frac{3}{(k+2)^2}$

7. $\displaystyle\sum_{k=2}^{\infty} \frac{7}{4k+2}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{e^k}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{(4+3k)^{7/6}}$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{1000k^2}{1+k^3}$

11. $\displaystyle\sum_{k=1}^{\infty} ke^{-3k^2}$

12. $\displaystyle\sum_{k=5}^{\infty} \frac{1000}{k(\ln k)^2}$

In Problems 13–22, use any test developed so far, including any from Section 9.2, to decide about the convergence or divergence of the series. Give a reason for your conclusion.

13. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+5}$

14. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{\pi}\right)^k$

15. $\displaystyle\sum_{k=1}^{\infty} \left[\left(\frac{1}{2}\right)^k + \frac{k-1}{2k+1}\right]$

16. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{2^k}\right)$

17. $\displaystyle\sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}\right)$

18. $\displaystyle\sum_{k=1}^{\infty} k \sin\frac{1}{k}$

19. $\displaystyle\sum_{k=1}^{\infty} k^2e^{-k^3}$

20. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

21. $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

22. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+4k^2}$

In Problems 23–26, estimate the error that is made by approximating the sum of the given series by the sum of the first five terms (see Example 5).

23. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^k}$

24. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$

25. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k^2}$

26. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

For the series given in Problems 27–32, determine how large n must be so that using the nth partial sum to approximate the series gives an error of no more than 0.0002.

27. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

28. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$

29. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k^2}$

30. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$

31. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1+k^4}$

32. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

33. For what values of p does $\sum_{n=2}^{\infty} 1/[n(\ln n)^p]$ converge? Explain.

34. Does $\sum_{n=3}^{\infty} 1/[n \cdot \ln n \cdot \ln(\ln n)]$ converge or diverge? Explain.

35. Use diagrams, as in Figure 1, to show that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

Hint: $\int_1^n (1/x)\,dx = \ln n$.

36. Using Problem 35, show that the sequence

$$B_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)$$

is increasing and bounded above by 1.

37. Use the result of Problem 35 to prove that $\lim_{n\to\infty} B_n$ in Problem 36 exists. (The limit, denoted γ, is called **Euler's constant** and is approximately 0.5772. It is currently not known whether γ is rational or irrational. It is known, however, that *if* γ is rational then the denominator in its reduced fraction is at least $10^{244,663}$.)

38. Use Problem 35 to get good upper and lower bounds for the sum of the first 10 million terms of the harmonic series.

39. From Problem 37, we infer that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \gamma + \ln(n+1)$$

Use this to estimate the number of terms of the harmonic series that are needed to get a sum greater than 20 and compare with the result reported in Problem 44 of Section 9.2.

40. Now that we have shown the existence of Euler's constant the hard way (Problems 35–37), we will solve a much more general problem the easy way and watch γ appear out of thin air, so to speak. Let f be continuous and decreasing on $[1, \infty)$ and let

$$B_n = f(1) + f(2) + \cdots + f(n) - \int_1^{n+1} f(x)\,dx$$

Note that B_n is the area of the shaded region in Figure 3.

(a) Why is it obvious that B_n increases with n?

(b) Show that $B_n \le f(1)$. *Hint:* Simply shift all the little shaded pieces leftward into the first rectangle.

(c) Conclude that $\lim_{n\to\infty} B_n$ exists.

(d) How do we get γ out of this?

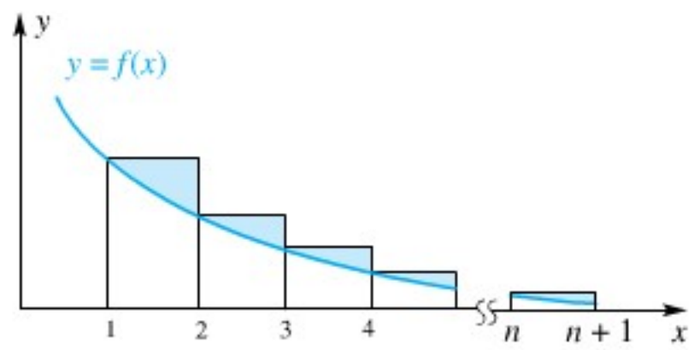


Figure 3

41. Let f be continuous, increasing, and concave down on $[1, \infty)$ as in Figure 4. Furthermore, let A_n be the area of the shaded region. Show that A_n is increasing with n, that $A_n \le T$ where T is the area of the outlined triangle, and thus that $\lim_{n\to\infty} A_n$ exists.

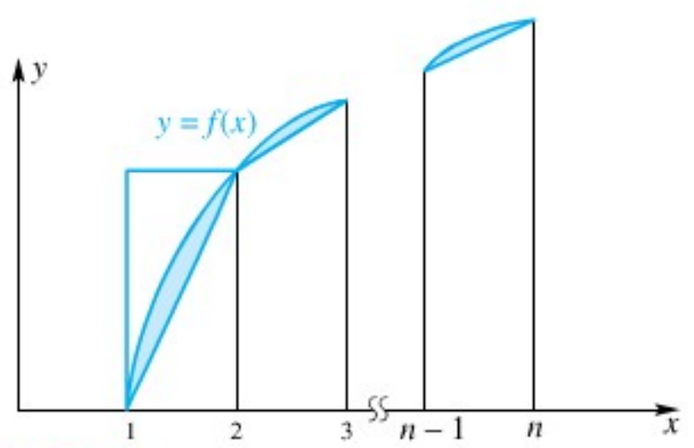


Figure 4

42. Specialize f of Problem 41 to $f(x) = \ln x$.

(a) Show that

$$A_n = \int_1^n \ln x\,dx - \left[\frac{\ln 1 + \ln 2}{2} + \cdots + \frac{\ln(n-1) + \ln n}{2}\right]$$
$$= n\ln n - n + 1 - \ln n! + \ln\sqrt{n}$$
$$= 1 + \ln\frac{(n/e)^n\sqrt{n}}{n!}$$

(b) Conclude from part (a) and Problem 41 that

$$k = \lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{n}}$$

exists. It can be shown that $k = \sqrt{2\pi}$.

(c) This means that $n! \approx \sqrt{2\pi n}(n/e)^n$, which is called **Stirling's Formula.** Use it to approximate 15! and compare it with the value that your calculator gives for 15!

43. Show that the error used in approximating S by S_n satisfies

$$E_n \ge \int_{n+1}^{\infty} f(x)\,dx$$

where the notation is the same as in the discussion preceding Example 5.

Answers to Concepts Review: **1.** bounded above **2.** $f(k)$; continuous; positive; nonincreasing **3.** convergence or divergence **4.** $p > 1$

9.4 Positive Series: Other Tests

We have completely analyzed the convergence and divergence of two series, the geometric series and the p-series.

$\sum_{n=1}^{\infty} r^n$ converges if $-1 < r < 1$, diverges otherwise

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p > 1, \text{ diverges otherwise}$$

In the first we have found what the series converges to, provided that it converges; in the second, we have not. These series provide standards, or models, against which we can measure other series. Keep in mind that we are still considering series whose terms are positive (or at least nonnegative).

Comparing One Series with Another A series with terms less than the corresponding terms of a convergent series ought to converge; a series with terms greater than the corresponding terms of a divergent series ought to diverge. What ought to be true is true.

Theorem A **Ordinary Comparison Test**

Suppose that $0 \le a_n \le b_n$ for $n \ge N$.

(i) If Σb_n converges, so does Σa_n.

(ii) If Σa_n diverges, so does Σb_n.

Proof We suppose that $N = 1$; the case $N > 1$ is only slightly harder. To prove (i), let $S_n = a_1 + a_2 + \cdots + a_n$ and note that $\{S_n\}$ is a nondecreasing sequence. If Σb_n converges, for instance, with sum B, then

$$S_n \le b_1 + b_2 + \cdots + b_n \le \sum_{n=1}^{\infty} b_n = B$$

By the Bounded Sum Test (Theorem 9.3A), Σa_n converges.

Property (ii) follows from (i); for if Σb_n converged, then Σa_n would have to converge. ■

EXAMPLE 1 Does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5n^2 - 4}$ converge or diverge?

SOLUTION A good guess would be that it diverges, since the nth term behaves like $1/5n$ for large n. In fact,

$$\frac{n}{5n^2 - 4} > \frac{n}{5n^2} = \frac{1}{5} \cdot \frac{1}{n}$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5} \cdot \frac{1}{n}$ diverges since it is one-fifth of the harmonic series (Theorem 9.2C). Thus, by the Ordinary Comparison Test, the given series also diverges. ■

EXAMPLE 2 Does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ converge or diverge?

SOLUTION A good guess would be that it converges, since the nth term behaves like $(1/2)^n$ for large n. To substantiate our guess, we note that

$$\frac{n}{2^n(n+1)} = \left(\frac{1}{2}\right)^n \frac{n}{n+1} < \left(\frac{1}{2}\right)^n$$

Since $\Sigma\left(\frac{1}{2}\right)^n$ converges (it is a geometric series with $r = \frac{1}{2}$), we conclude that the given series converges. ■

If there is a problem in applying the Ordinary Comparison Test, it is in finding exactly the right known series with which to compare the series to be tested. Suppose that we wish to determine the convergence or divergence of

$$\sum_{n=3}^{\infty} \frac{1}{(n-2)^2} = \sum_{n=3}^{\infty} \frac{1}{n^2 - 4n + 4}$$

We suspect convergence, so our inclination is to compare $1/(n-2)^2$ with $1/n^2$, but, unfortunately,

$$\frac{1}{(n-2)^2} > \frac{1}{n^2}$$

which gives no test at all (the inequality goes the wrong way for what we want). After some experimenting, we discover that

$$\frac{1}{(n-2)^2} \leq \frac{9}{n^2}$$

for $n \geq 3$; since $\Sigma 9/n^2$ converges, so does $\Sigma 1/(n-2)^2$.

Can we avoid these contortions with inequalities? Our intuition tells us that Σa_n and Σb_n converge or diverge together, provided that a_n and b_n are approximately the same size for large n (give or take a multiplicative constant). This is the essential content of our next theorem.

Theorem B **Limit Comparison Test**

Suppose that $a_n \geq 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

If $0 < L < \infty$, then Σa_n and Σb_n converge or diverge together. If $L = 0$ and Σb_n converges, then Σa_n converges.

Proof Begin by taking $\varepsilon = L/2$ in the definition of limit of a sequence (Section 9.1). There is a number N such that $n \geq N \Rightarrow |(a_n/b_n) - L| < L/2$; that is,

$$-\frac{L}{2} < \frac{a_n}{b_n} - L < \frac{L}{2}$$

This inequality is equivalent (by adding L throughout) to

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$$

Hence, for $n \geq N$,

$$b_n < \frac{2}{L} a_n \quad \text{and} \quad a_n < \frac{3L}{2} b_n$$

These two inequalities, together with the Ordinary Comparison Test, show that Σa_n and Σb_n converge or diverge together. We leave the proof of the final statement of the theorem to the reader (Problem 37). ■

EXAMPLE 3 Determine the convergence or divergence of each series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{n^3 - 2n^2 + 11}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 19n}}$

SOLUTION We apply the Limit Comparison Test, but we still must decide to what we should compare the nth term. We see what the nth term is like for large n by looking at the largest-degree terms in the numerator and denominator. In the first case, the nth term is like $3/n^2$; in the second, it is like $1/n$.

(a) $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n-2)/(n^3 - 2n^2 + 11)}{3/n^2} = \lim_{n\to\infty} \frac{3n^3 - 2n^2}{3n^3 - 6n^2 + 33} = 1$$

(b) $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2 + 19n}}{1/n} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2 + 19n}} = 1$$

Since $\Sigma 3/n^2$ converges and $\Sigma 1/n$ diverges, we conclude that the series in (a) converges and the series in (b) diverges. ■

EXAMPLE 4 Does $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ converge or diverge?

SOLUTION To what shall we compare $(\ln n)/n^2$? If we try $1/n^2$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \div \frac{1}{n^2} = \lim_{n\to\infty} \ln n = \infty$$

The test fails because its conditions are not satisfied. On the other hand, if we use $1/n$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \div \frac{1}{n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Again, the test fails. Possibly something between $1/n^2$ and $1/n$ will work, such as $1/n^{3/2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \div \frac{1}{n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$$

(The last equality follows from l'Hôpital's Rule.) We conclude from the second part of the Limit Comparison Test that $\Sigma(\ln n)/n^2$ converges (since $\Sigma 1/n^{3/2}$ converges by the p-Series Test). ■

Comparing a Series with Itself Getting useful results from the comparison tests requires insight or perseverance. We must choose wisely among known series to find one that is just right for comparison with the series that we wish to test. Wouldn't it be nice if we could somehow compare a series with itself and thereby determine convergence or divergence? Roughly speaking, this is what we do in the Ratio Test.

Theorem C Ratio Test

Let Σa_n be a series of positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

(i) If $\rho < 1$, the series converges.

(ii) If $\rho > 1$ or if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$, the series diverges.

(iii) If $\rho = 1$, the test is inconclusive.

Proof Here is what is behind the Ratio Test. Since $\lim_{n\to\infty} a_{n+1}/a_n = \rho$, $a_{n+1} \approx \rho a_n$; that is, the series behaves like a geometric series with ratio ρ. A geometric series converges when its ratio is less than 1, and diverges when its ratio is greater than 1. Tying down this argument is the task before us.

(i) Since $\rho < 1$, we may choose a number r such that $\rho < r < 1$, for example, $r = (\rho + 1)/2$. Next choose N so large that $n \geq N$ implies that $a_{n+1}/a_n < r$. (This can be done since $\lim_{n\to\infty} a_{n+1}/a_n = \rho < r$.)

Then

$$\begin{aligned} a_{N+1} &< ra_N \\ a_{N+2} &< ra_{N+1} < r^2 a_N \\ a_{N+3} &< ra_{N+2} < r^3 a_N \\ &\vdots \end{aligned}$$

Since $ra_N + r^2a_N + r^3a_N + \cdots$ is a geometric series with $0 < r < 1$, it converges. By the Ordinary Comparison Test, $\sum_{n=N+1}^{\infty} a_n$ converges, and hence so does $\sum_{n=1}^{\infty} a_n$.

(ii) Since $\rho > 1$, there is a number N such that $a_{n+1}/a_n > 1$ for all $n \geq N$. Thus,

$$a_{N+1} > a_N$$

$$a_{N+2} > a_{N+1} > a_N$$

$$\vdots$$

Hence, $a_n > a_N > 0$ for all $n > N$, which means that $\lim_{n\to\infty} a_n$ cannot be zero. By the nth-Term Test for Divergence, Σa_n diverges.

(iii) We know that $\Sigma 1/n$ diverges, whereas $\Sigma 1/n^2$ converges. For the first series,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

For the second series,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \div \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Thus, the Ratio Test does not distinguish between convergence and divergence when $\rho = 1$. ■

The Ratio Test will always be inconclusive for any series whose nth term is a rational expression in n, since in this case $\rho = 1$ (the cases $a_n = 1/n$ and $a_n = 1/n^2$ were treated above). However, for a series whose nth term involves $n!$ or r^n, the Ratio Test usually works beautifully.

EXAMPLE 5 Test for convergence or divergence: $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

SOLUTION

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

We conclude by the Ratio Test that the series converges. ■

EXAMPLE 6 Test for convergence or divergence: $\sum_{n=1}^{\infty} \frac{2^n}{n^{20}}$.

SOLUTION

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^{20}} \frac{n^{20}}{2^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{20} \cdot 2 = 2$$

We conclude that the given series diverges. ■

EXAMPLE 7 Test for convergence or divergence: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

SOLUTION We will need the fact that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{1/h} = e$$

which follows from Theorem 6.5A. Taking this as known, we may write

$$\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\frac{1}{((n+1)/n)^n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$$

Therefore, the given series converges. ■

Summary To test a series Σa_n of positive terms for convergence or divergence, look carefully at a_n.

1. If $\lim_{n\to\infty} a_n \neq 0$, conclude from the nth-Term Test that the series diverges.
2. If a_n involves $n!$, r^n, or n^n, try the Ratio Test.
3. If a_n involves only constant powers of n, try the Limit Comparison Test. In particular, if a_n is a rational expression in n, use this test with b_n as the quotient of the leading terms from the numerator and denominator.
4. If the tests above do not work, try the Ordinary Comparison Test, the Integral Test, or the Bounded Sum Test.
5. Some series require a clever manipulation or a neat trick to determine convergence or divergence.

Concepts Review

1. The Ordinary Comparison Test says that if _____ and if Σb_k converges, then Σa_k also converges.

2. Assume that $a_k \geq 0$ and $b_k > 0$. The Limit Comparison Test says that if $0 <$ _____ $< \infty$ then Σa_k and Σb_k converge or diverge together.

3. Let $\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$. The Ratio Test says that a series Σa_k of positive terms converges if _____, diverges if _____, and may do either if _____.

4. $\Sigma(3^k/k!)$ is an obvious candidate for the _____ Test, whereas $\Sigma k/(k^3 - k - 1)$ is an obvious candidate for the _____ Test.

Problem Set 9.4

In Problems 1–4, use the Limit Comparison Test to determine convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+2n+3}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{3n+1}{n^3-4}$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n+1}}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2n+1}}{n^2}$

In Problems 5–10, use the Ratio Test to determine convergence or divergence.

5. $\displaystyle\sum_{n=1}^{\infty}\frac{8^n}{n!}$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{n^5}$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^{100}}$

8. $\displaystyle\sum_{n=1}^{\infty} n\left(\tfrac{1}{3}\right)^n$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{(2n)!}$

10. $\displaystyle\sum_{k=1}^{\infty}\frac{3^k+k}{k!}$

In Problems 11–34, determine convergence or divergence for each of the series. Indicate the test you use.

11. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n+200}$

12. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{5+n}$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{n+3}{n^2\sqrt{n}}$

14. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n^2+1}$

15. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n!}$

16. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{2^n}$

17. $\displaystyle\sum_{n=1}^{\infty}\frac{4n^3+3n}{n^5-4n^2+1}$

18. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{3^n}$

19. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\cdots$

Hint: $a_n=\dfrac{1}{n(n+1)}$.

20. $\dfrac{1}{2^2}+\dfrac{2}{3^2}+\dfrac{3}{4^2}+\dfrac{4}{5^2}+\cdots$

21. $\dfrac{2}{1\cdot 3\cdot 4}+\dfrac{3}{2\cdot 4\cdot 5}+\dfrac{4}{3\cdot 5\cdot 6}+\dfrac{5}{4\cdot 6\cdot 7}+\cdots$

22. $\dfrac{1}{1^2+1}+\dfrac{2}{2^2+1}+\dfrac{3}{3^2+1}+\dfrac{4}{4^2+1}+\cdots$

23. $\dfrac{1}{3}+\dfrac{2}{3^2}+\dfrac{3}{3^3}+\dfrac{4}{3^4}+\cdots$ **24.** $3+\dfrac{3^2}{2!}+\dfrac{3^3}{3!}+\dfrac{3^4}{4!}+\cdots$

25. $1+\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}+\dfrac{1}{4\sqrt{4}}+\cdots$

26. $\dfrac{\ln 2}{2^2}+\dfrac{\ln 3}{3^2}+\dfrac{\ln 4}{4^2}+\dfrac{\ln 5}{5^2}+\cdots$

27. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sin^2 n}$ **28.** $\displaystyle\sum_{n=1}^{\infty}\frac{5}{3^n+1}$

29. $\displaystyle\sum_{n=1}^{\infty}\frac{4+\cos n}{n^3}$ **30.** $\displaystyle\sum_{n=1}^{\infty}\frac{5^{2n}}{n!}$

31. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(2n)!}$ **32.** $\displaystyle\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)^n$

33. $\displaystyle\sum_{n=1}^{\infty}\frac{4^n+n}{n!}$ **34.** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2+n5^n}$

35. Let $a_n>0$ and suppose that Σa_n converges. Prove that Σa_n^2 converges.

36. Prove that $\lim_{n\to\infty}(n!/n^n)=0$ by considering the series $\Sigma n!/n^n$. *Hint:* Example 7, followed by nth-Term Test.

37. Prove that if $a_n\ge 0, b_n>0, \lim_{n\to\infty} a_n/b_n=0$, and Σb_n converges then Σa_n converges.

38. Prove that if $a_n\ge 0, b_n>0, \lim_{n\to\infty} a_n/b_n=\infty$, and Σb_n diverges then Σa_n diverges.

39. Suppose that $\lim_{n\to\infty} na_n=1$. Prove that Σa_n diverges.

40. Prove that if Σa_n is a convergent series of positive terms then $\Sigma\ln(1+a_n)$ converges.

41. Root Test Prove that if $a_n>0$ and $\lim_{n\to\infty}(a_n)^{1/n}=R$ then Σa_n converges if $R<1$ and diverges if $R>1$.

42. Test for convergence or divergence using the Root Test.

(a) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{3n+2}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}+\frac{1}{n}\right)^n$

43. Test for convergence or divergence. In some cases, a clever manipulation using the properties of logarithms will simplify the problem.

(a) $\displaystyle\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$ (b) $\displaystyle\sum_{n=1}^{\infty}\ln\left[\frac{(n+1)^2}{n(n+2)}\right]$

(c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ (d) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{[\ln(\ln n)]^{\ln n}}$

(e) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\ln n)^4}$ (f) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\ln n}{n}\right]^2$

EXPL **44.** Let $p(n)$ and $q(n)$ be polynomials in n with nonnegative coefficients. Give simple conditions that determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}$.

EXPL **45.** Give conditions on p that determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)$.

46. Test for convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ (b) $\displaystyle\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty}\sqrt{n}\left[1-\cos\left(\frac{1}{n}\right)\right]$

Answers to Concepts Review: **1.** $0\le a_k\le b_k$ **2.** $\lim_{k\to\infty}(a_k/b_k)$ **3.** $\rho<1;\ \rho>1;\ \rho=1$ **4.** Ratio: Limit Comparison

9.5 Alternating Series, Absolute Convergence, and Conditional Convergence

In the last two sections, we considered series of nonnegative terms. Now we remove that restriction, allowing some terms to be negative. In particular, we study **alternating series,** that is, series of the form

$$a_1-a_2+a_3-a_4+\cdots$$

where $a_n>0$ for all n. An important example is the **alternating harmonic series**

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots$$

We have seen that the harmonic series diverges; we shall soon see that the *alternating* harmonic series converges.

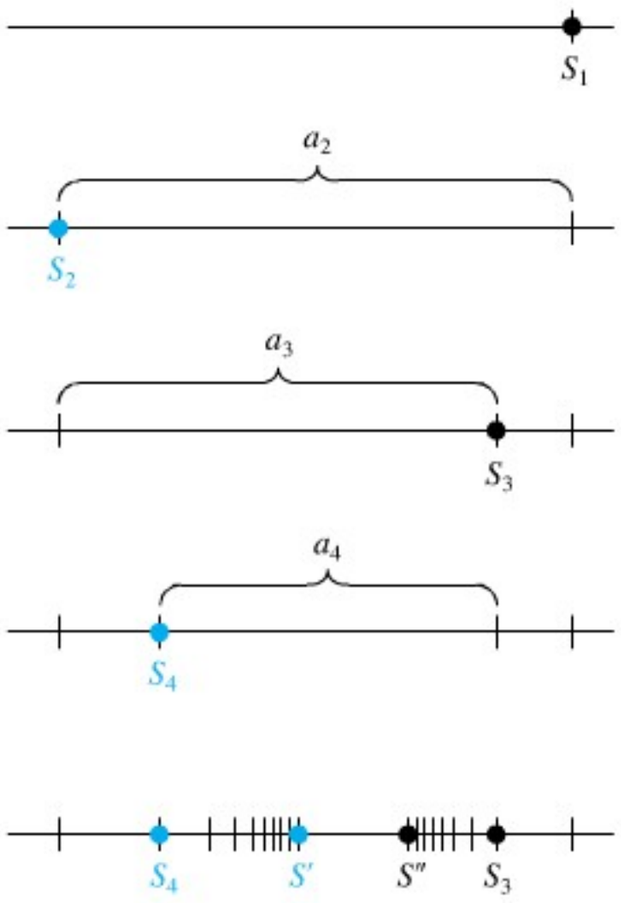


Figure 1

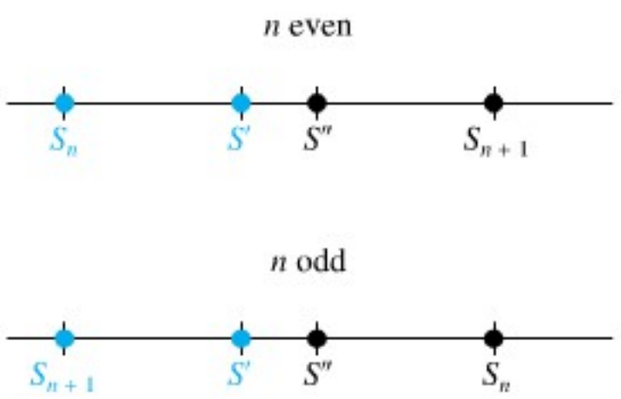


Figure 2

A Convergence Test Let us suppose that the sequence $\{a_n\}$ is decreasing; that is, $a_{n+1} < a_n$ for all n. Also, let S_n have its usual meaning. Thus, for the alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, we have

$$S_1 = a_1$$
$$S_2 = a_1 - a_2 = S_1 - a_2$$
$$S_3 = a_1 - a_2 + a_3 = S_2 + a_3$$
$$S_4 = a_1 - a_2 + a_3 - a_4 = S_3 - a_4$$

and so on. A geometric interpretation of these partial sums is shown in Figure 1. Note that the even-numbered terms $S_2, S_4, S_6, \ldots$ are increasing and bounded above and hence must converge to a limit, call it S'. Similarly, the odd-numbered terms $S_1, S_3, S_5, \ldots$ are decreasing and bounded below. They also converge, say to S''.

Both S' and S'' are between S_n and S_{n+1} for all n (see Figure 2), and so

$$|S'' - S'| \leq |S_{n+1} - S_n| = a_{n+1}$$

Thus, the condition $a_{n+1} \to 0$ as $n \to \infty$ will guarantee that $S' = S''$ and, consequently, the convergence of the series to their common value, which we call S. Finally, we note that, since S is between S_n and S_{n+1},

$$|S - S_n| \leq |S_{n+1} - S_n| = a_{n+1}$$

That is, the error made by using S_n as an approximation to the sum S of the whole series is not more than the magnitude of the first neglected term. We have proved the following theorem.

Theorem A Alternating Series Test

Let

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

be an alternating series with $a_n > a_{n+1} > 0$. If $\lim_{n\to\infty} a_n = 0$, then the series converges. Moreover, the error made by using the sum S_n of the first n terms to approximate the sum S of the series is not more than a_{n+1}.

EXAMPLE 1 Show that the alternating harmonic series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

converges. How many terms of this series would we need to take in order to get a partial sum S_n within 0.01 of the sum S of the whole series?

SOLUTION The alternating harmonic series satisfies the hypotheses of Theorem A and so converges. We want $|S - S_n| \leq 0.01$, and this will hold if $a_{n+1} \leq 0.01$. Since $a_{n+1} = 1/(n+1)$, we require $1/(n+1) \leq 0.01$, which is satisfied if $n \geq 99$. Thus, we need to take 99 terms to make sure that we have the desired accuracy. This gives you an idea of how slowly the alternating harmonic series converges. (See Problem 45 for a clever way to find the exact sum of this series.) ■

EXAMPLE 2 Show that

$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

converges. Calculate S_5 and estimate the error made by using this as a value for the sum of the whole series.

SOLUTION The Alternating Series Test (Theorem A) applies and guarantees convergence.

$$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$$

$$|S - S_5| \leq a_6 = \frac{1}{6!} \approx 0.0014$$ ■

EXAMPLE 3 Show that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{2^n}$ converges.

SOLUTION To get a feeling for this series, we write the first few terms:

$$\tfrac{1}{2} - 1 + \tfrac{9}{8} - 1 + \tfrac{25}{32} - \tfrac{36}{64} + \cdots$$

The series is alternating and $\lim_{n\to\infty} n^2/2^n = 0$ (l'Hôpital's Rule), but unfortunately the terms are not decreasing initially. However, they do appear to be decreasing after the first two terms; this is good enough, since what happens at the beginning of a series never affects convergence or divergence. To show that the sequence $\{n^2/2^n\}$ is decreasing from the third term on, consider the function

$$f(x) = \frac{x^2}{2^x}$$

Note that if $x \geq 3$ the derivative

$$f'(x) = \frac{2x \cdot 2^x - x^2 2^x \ln 2}{2^{2x}} = \frac{x2^x(2 - x \ln 2)}{2^{2x}}$$

$$\approx \frac{x(2 - 0.69x)}{2^x} < 0$$

Thus, f is decreasing on $[3, \infty)$, and so $\{n^2/2^n\}$ is decreasing for $n \geq 3$. For a different demonstration of this last fact, see Example 6 of Section 9.1. ■

Absolute Convergence

Does a series such as

$$1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{1}{36} + \cdots$$

in which there is a pattern of two positive terms followed by one negative term, converge or diverge? The Alternating Series Test does not apply. However, since the corresponding series of all positive terms

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \tfrac{1}{36} + \cdots$$

converges (p-series with $p = 2$), it seems plausible to think that the same series with some terms negative should converge (even better). This is the content of our next theorem.

Theorem B Absolute Convergence Test

If $\Sigma|u_n|$ converges, then Σu_n converges.

Proof We use a trick. Let $v_n = u_n + |u_n|$, so

$$u_n = v_n - |u_n|$$

Now $0 \leq v_n \leq 2|u_n|$, and so Σv_n converges by the Ordinary Comparison Test. It follows from the Linearity Theorem (Theorem 9.2B) that $\Sigma u_n = \Sigma(v_n - |u_n|)$ converges. ■

A series Σu_n is said to **converge absolutely** if $\Sigma |u_n|$ converges. Theorem B asserts that absolute convergence implies convergence. All our tests for convergence of series of positive terms are automatically tests for the absolute convergence of a series in which some terms are negative. In particular, this is true of the Ratio Test, which we now restate.

Theorem C **Absolute Ratio Test**

Let Σu_n be a series of nonzero terms and suppose that

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \rho$$

(i) If $\rho < 1$, the series converges absolutely (hence converges).
(ii) If $\rho > 1$, the series diverges.
(iii) If $\rho = 1$, the test is inconclusive.

Proof Proofs of (i) and (iii) are direct results of the Ratio Test. For (ii), we could conclude from the original Ratio Test that $\Sigma |u_n|$ diverges, but here we are claiming more, that Σu_n diverges. Since

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} > 1$$

it follows that for n sufficiently large, say $n \geq N$, $|u_{n+1}| > |u_n|$. This, in turn, implies that $|u_n| > |u_N| > 0$ for all $n \geq N$, and so $\lim_{n\to\infty} u_n$ cannot be 0. We conclude by the nth-Term Test that Σu_n diverges. ■

EXAMPLE 4 Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!}$ converges absolutely.

SOLUTION

$$\rho = \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \div \frac{3^n}{n!}$$

$$= \lim_{n\to\infty} \frac{3}{n+1} = 0$$

We conclude from the Absolute Ratio Test that the series converges absolutely (and therefore converges). ■

EXAMPLE 5 Test for the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\cos(n!)}{n^2}$.

SOLUTION If you write out the first 100 terms of this series, you will discover that the signs of the terms vary in a rather random way. The series is in fact a difficult one to analyze directly. However,

$$\left| \frac{\cos(n!)}{n^2} \right| \leq \frac{1}{n^2}$$

and so the series converges absolutely by the Ordinary Comparison Test. We conclude from the Absolute Convergence Test (Theorem B) that the series converges. ■

Conditional Convergence A common error is to try to turn Theorem B around. It does *not* say that convergence implies absolute convergence. That is clearly false; witness the alternating harmonic series. We know that

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

converges, but that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

diverges. A series Σu_n is called **conditionally convergent** if Σu_n converges but $\Sigma |u_n|$ diverges. The alternating harmonic series is the premier example of a conditionally convergent series, but there are many others.

EXAMPLE 6 Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ is conditionally convergent.

SOLUTION $\sum_{n=1}^{\infty} (-1)^{n+1} [1/\sqrt{n}\,]$ converges by the Alternating Series Test. However, $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges, since it is a p-series with $p = \frac{1}{2}$. ■

Absolutely convergent series behave much better than do conditionally convergent ones. Here is a nice theorem about absolutely convergent series. It is spectacularly false for conditionally convergent series (see Problems 35–38). The proof is difficult, so we do not include it here.

Theorem D Rearrangement Theorem

The terms of an absolutely convergent series can be rearranged without affecting either the convergence or the sum of the series.

For example, the series

$$1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{1}{36} + \tfrac{1}{49} + \tfrac{1}{64} - \tfrac{1}{81} + \cdots$$

converges absolutely. The rearrangement

$$1 + \tfrac{1}{4} + \tfrac{1}{16} - \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \tfrac{1}{64} - \tfrac{1}{36} + \cdots$$

converges and has the same sum as the original series.

Concepts Review

1. If $a_n \geq 0$ for all n, the alternating series $a_1 - a_2 + a_3 - \cdots$ will converge provided that the terms are decreasing in size and _____.

2. If $\Sigma |u_k|$ converges, we say that the series Σu_k converges _____; if Σu_k converges, but $\Sigma |u_k|$ diverges, we say that Σu_k converges _____.

3. The premier example of a conditionally convergent series is _____.

4. The terms of an absolutely convergent series may be _____ at will without affecting its convergence or its sum.

Problem Set 9.5

In Problems 1–6, show that each alternating series converges, and then estimate the error made by using the partial sum S_9 as an approximation to the sum S of the series (see Examples 1–3).

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{3n+1}$

2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

5. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$

In Problems 7–12, show that each series converges absolutely.

7. $\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\sqrt{n}}$

9. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

10. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{e^n}$

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$

12. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$

In Problems 13–30, classify each series as absolutely convergent, conditionally convergent, or divergent.

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n}$

14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n^{1.1}}$

15. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{10n+1}$

16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{10n^{1.1}+1}$

17. $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{n\ln n}$

18. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n(1+\sqrt{n})}$

19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^4}{2^n}$

20. $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\sqrt{n^2-1}}$

21. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$

22. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n-1}{n}$

23. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$

24. $\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}$

25. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n\sqrt{n}}$

26. $\sum_{n=1}^{\infty}n\sin\left(\frac{1}{n}\right)$

27. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n(n+1)}}$

28. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$

29. $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{n^2}$

30. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{\pi}{n}$

31. Prove that if Σa_n diverges, so does $\Sigma|a_n|$.

32. Give an example of two series Σa_n and Σb_n, both convergent, such that $\Sigma a_n b_n$ diverges.

33. Show that the positive terms of the alternating harmonic series form a divergent series. Show the same for the negative terms.

34. Show that the results in Problem 33 hold for any conditionally convergent series.

35. Show that the alternating harmonic series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\cdots$$

(whose sum is actually $\ln 2 \approx 0.69$) can be rearranged to converge to 1.3 by using the following steps.

(a) Take enough of the positive terms $1+\frac{1}{3}+\frac{1}{5}+\cdots$ to just exceed 1.3.

(b) Now add enough of the negative terms $-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots$ so that the partial sum S_n falls just below 1.3.

(c) Add just enough more positive terms to again exceed 1.3, and so on.

C **36.** Use your calculator to help you find the first 20 terms of the series described in Problem 35. Calculate S_{20}.

37. Explain why a conditionally convergent series can be rearranged to converge to any given number.

38. Show that a conditionally convergent series can be rearranged so as to diverge.

39. Show that $\lim_{n\to\infty} a_n = 0$ is not sufficient to guarantee the convergence of the alternating series $\Sigma(-1)^{n+1}a_n$. *Hint:* Alternate the terms of $\Sigma 1/n$ and $\Sigma(-1/n^2)$.

40. Discuss the convergence or divergence of

$$\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}-1}-\frac{1}{\sqrt{4}+1}+\cdots$$

41. Prove that if $\sum_{k=1}^{\infty}a_k^2$ and $\sum_{k=1}^{\infty}b_k^2$ both converge then $\sum_{k=1}^{\infty}a_k b_k$ converges absolutely. *Hint:* First show that $2|a_k b_k| \le a_k^2 + b_k^2$.

42. Sketch the graph of $y = (\sin x)/x$ and then show that $\int_0^{\infty}(\sin x)/x\,dx$ converges.

43. Show that $\int_0^{\infty}|\sin x|/x\,dx$ diverges.

44. Show that the graph of $y = x\sin\frac{\pi}{x}$ on $(0, 1]$ has infinite length.

45. Note that

$$\begin{aligned}1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n} &= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\\ &= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\end{aligned}$$

Recognize the latter expression as a Riemann sum and use it to find the sum of the alternating harmonic series.

Answers to Concepts Review: **1.** $\lim_{n\to\infty} a_n = 0$ **2.** absolutely; conditionally **3.** the alternating harmonic series **4.** rearranged