Outline

- What is transformation
- 2D transformations
- 3D transformations

What is transforms?

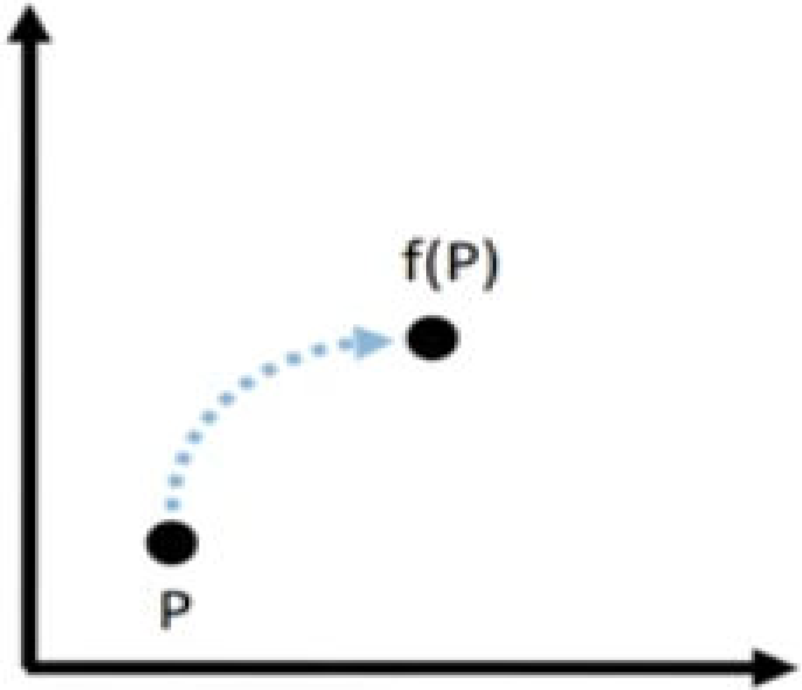
➤ Operations that are applied to the geometric description of an object to change its position, orientation or size.

Basic transformation:

- Translation
- Rotation
- Scaling

What Are Transforms?

- Just functions acting on points
  - $(x',y',z') = f(x,y,z)$
  - $P' = f(P)$



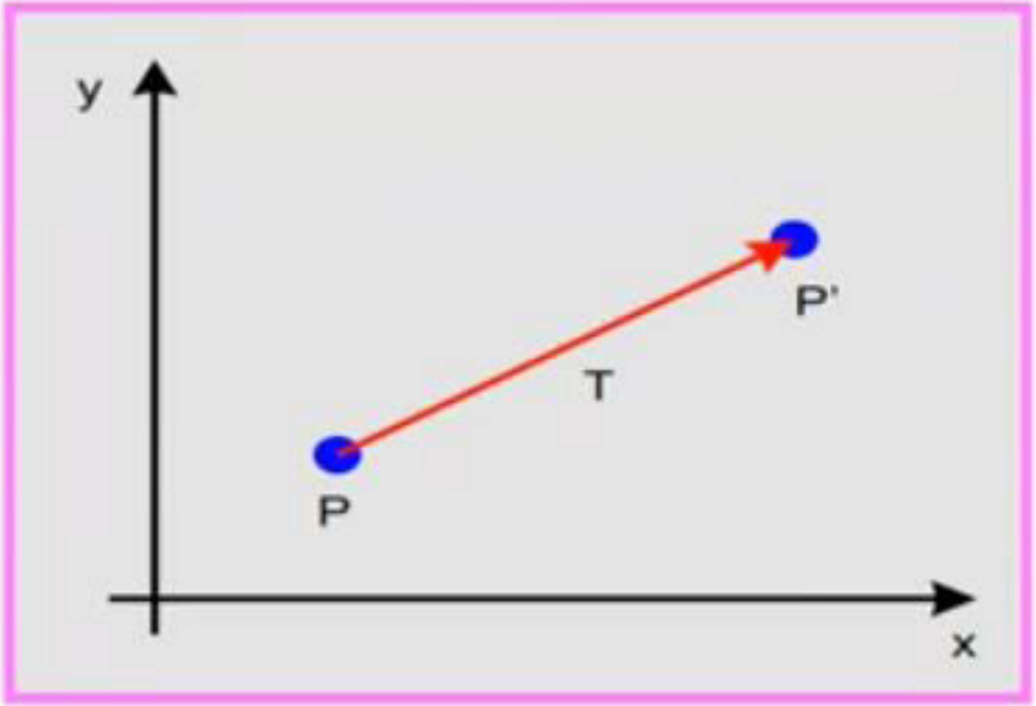
Outline

- What is transformation
- 2D transformations
- 3D transformations

2D Translation

Translation

- **2D Translation:** Move a point along a straight-line path to its new location.

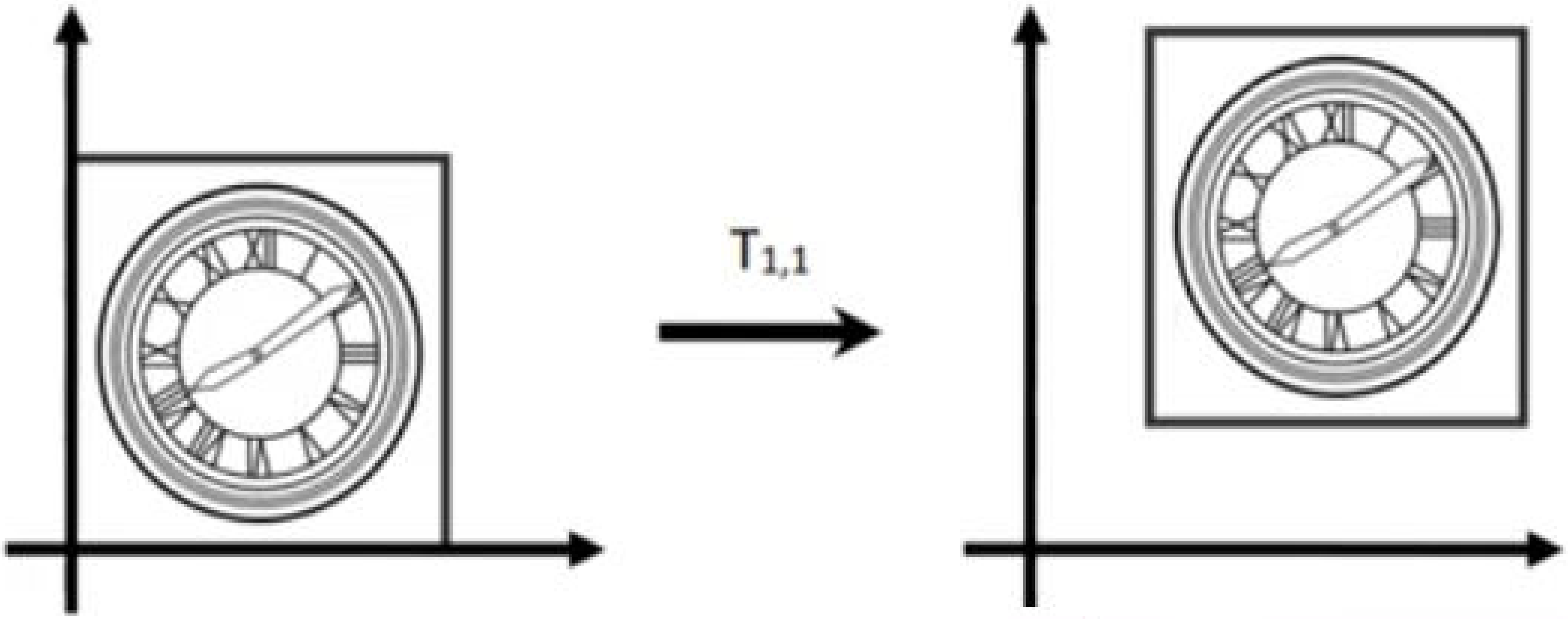


$$x' = x + t_x, \qquad y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = P + T$$

Translation?



$$x' = x + t_x$$
$$y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix}$$

2D translation Example

Q. Translate a polygon with coordinates A (2,7), B(7,10), C(10,2) by 3 units in X direction and 4 units in Y direction

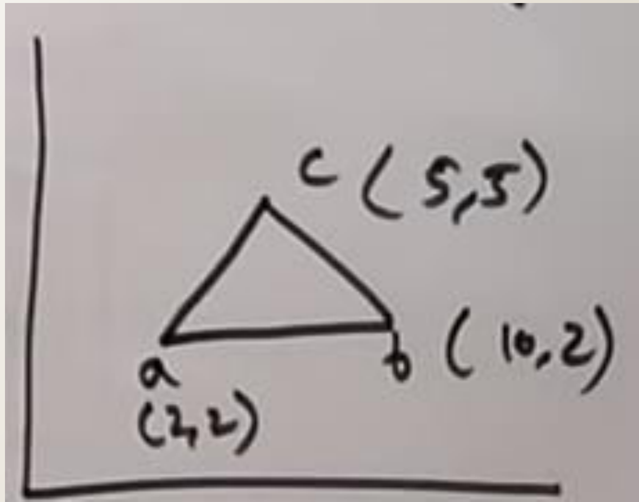
$A' \equiv (5, 11)$

$B' \equiv (10, 14)$

$C' \equiv (13, 6)$

2D translation Example

a(2,2), b(10,2) c(5,5), Translate the triangle with dx=5, dy=6



Answer

$a(2,2) \quad b(10,2) \quad c(5,5)$

$$A' = A + T$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$a' = (7, 8)$$

$$B' = B + T$$

$$= \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \end{bmatrix}$$

$$b' = (15, 8)$$

$$c' = c + T$$

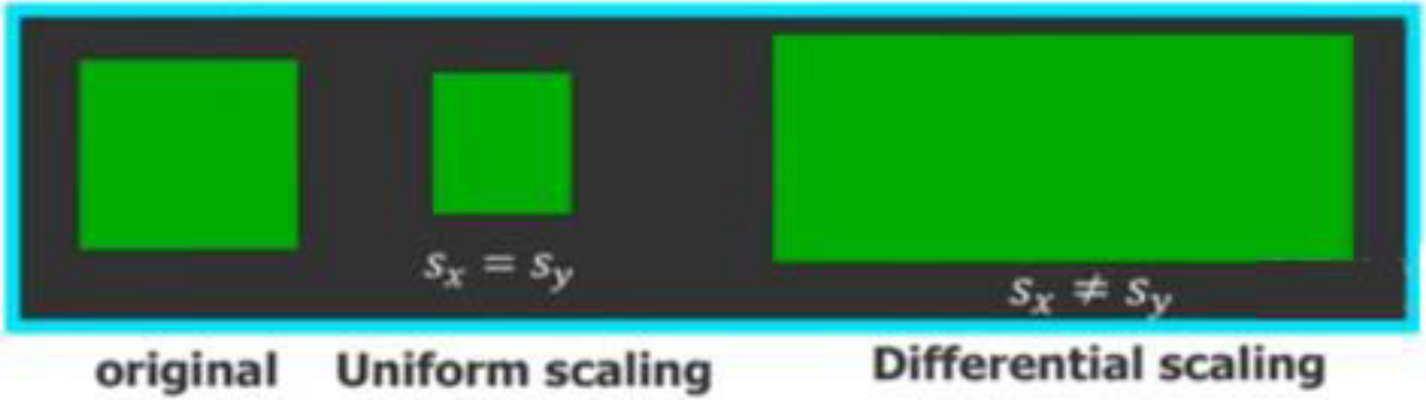
$$= \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$c' = (10, 11)$$

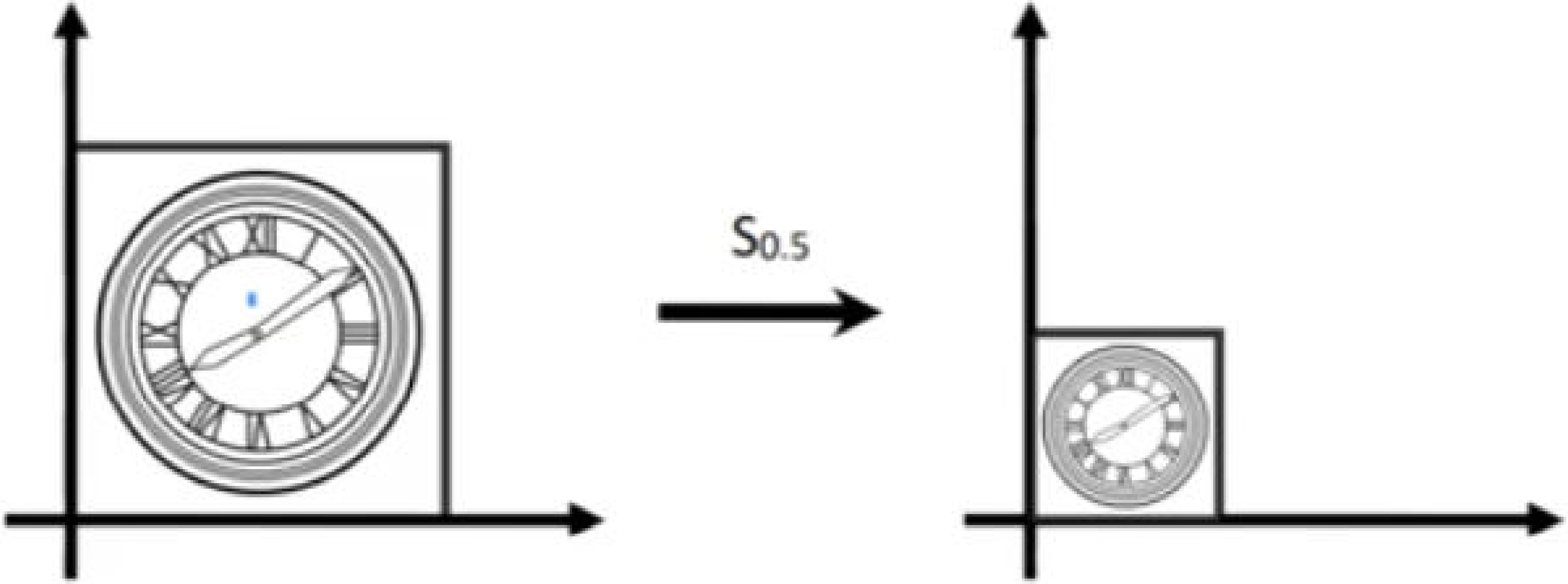
2D Scaling

2D Scaling

- A positive numeric values can be assigned to the scaling factors.
- Values less than 1 reduce the size of objects, and greater than 1 produce an enlargement.
- ***Uniform Scaling:*** $s_x = s_y$
- ***Differential Scaling:*** $s_x \neq s_y$, used in modeling applications.



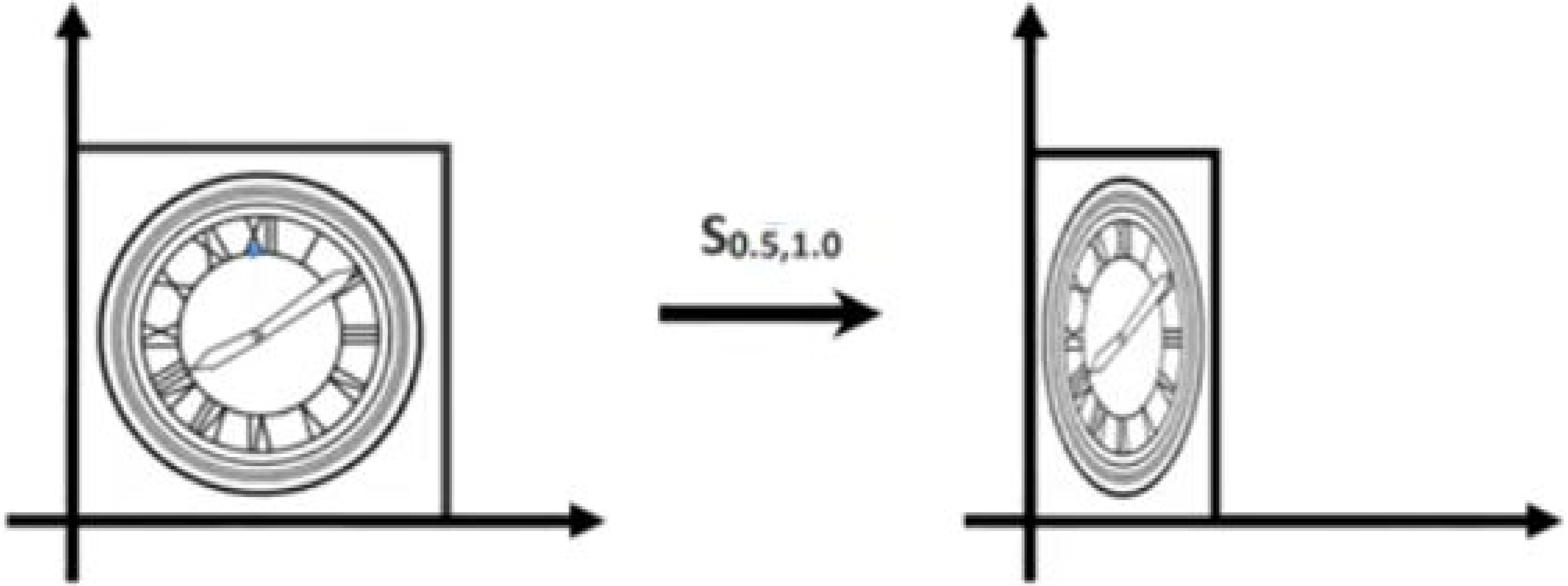
Scale Matrix (Uniform)



$$x' = s.x$$
$$y' = s.y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale (Non-Uniform)



$$x' = s_x \cdot x$$
$$y' = s_y \cdot y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D scaling Example 1

1. Scale a polygon with coordinates A (2,5), B (7,10), C (10,2) by 2 units in X-direction & 3 units in Y-direction.

→ A (2,5)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}$$

⇒ A' (4, 15)

→ B (7,10)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 14 \\ 30 \end{bmatrix}$$

⇒ B' (14,30)

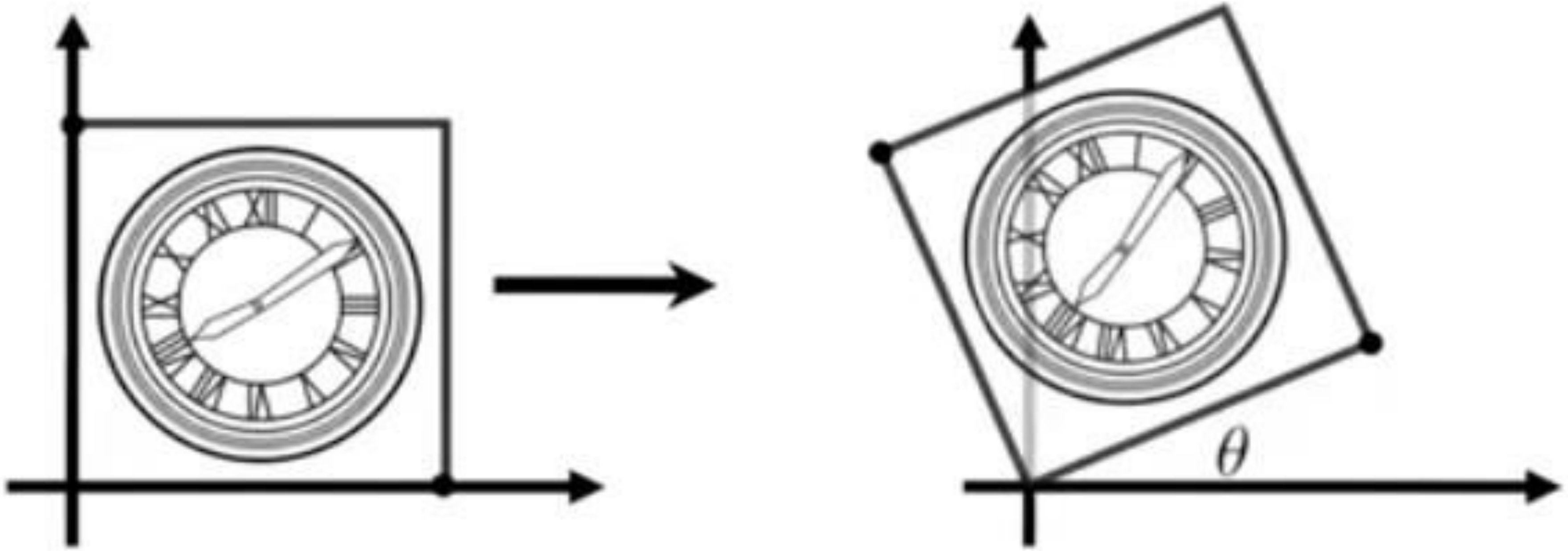
→ C (10, 2)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \end{bmatrix}$$

⇒ C' (20, 6)

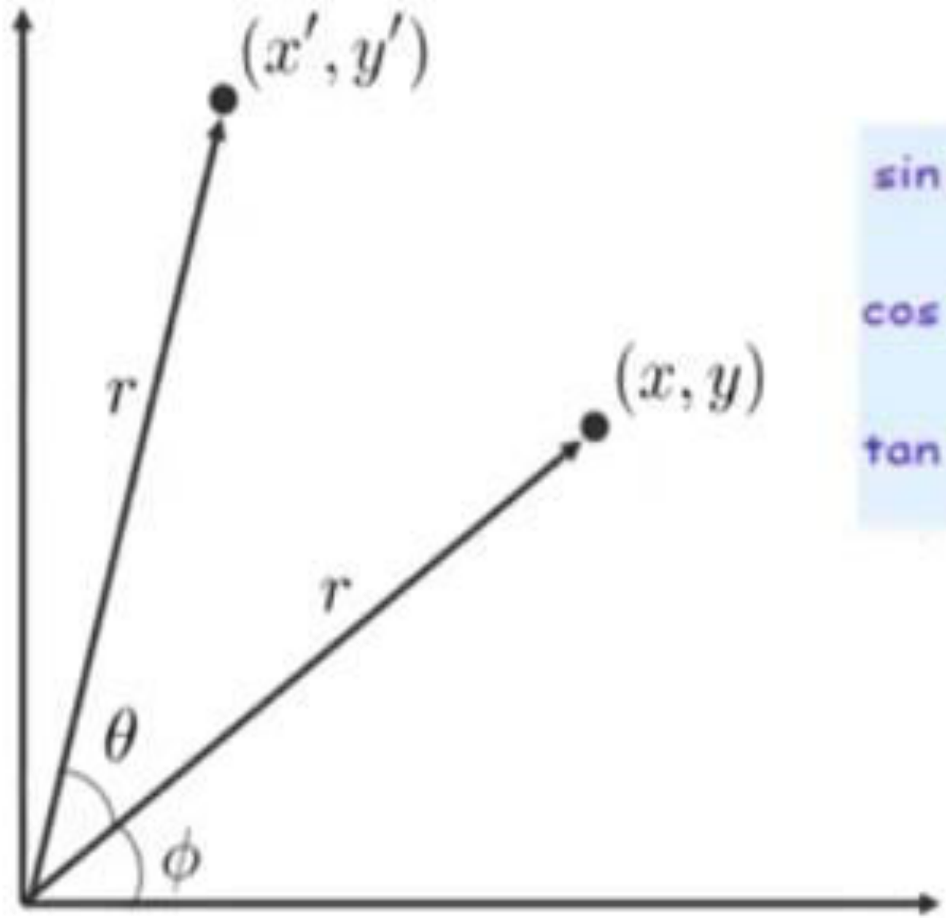
2D Rotation

Rotation Matrix



2-D Rotation

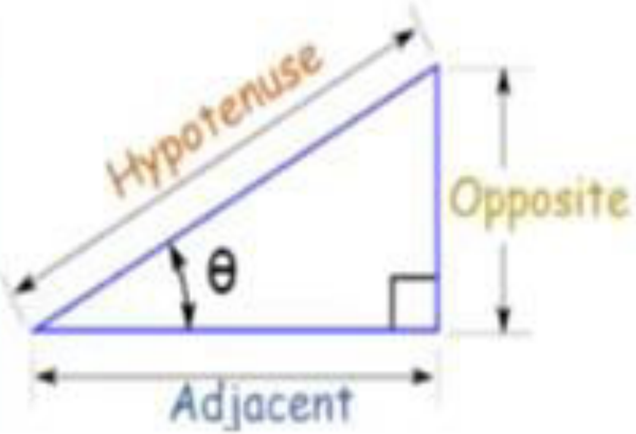
- Sine, Cosine and Tangent are the main functions used in Trigonometry



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$



https://www.mathsisfun.com/geometry/unit-circle.html

2D Rotation

$$\cos\phi = \frac{x}{r} \qquad x = r\cos\phi$$

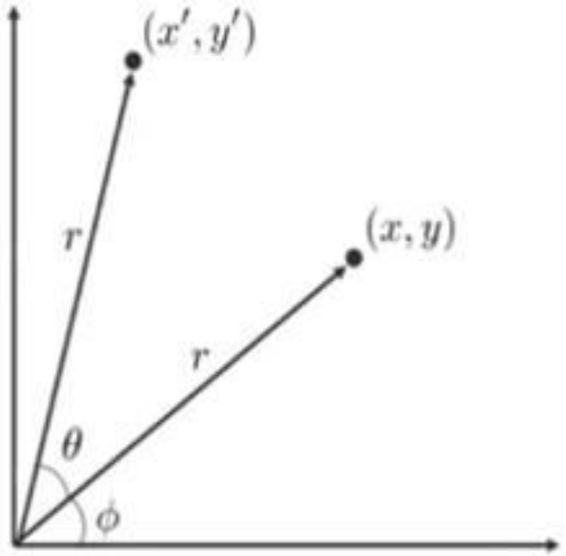
$$\sin\phi = \frac{y}{r} \qquad y = r\sin\phi$$

$$x' = r\cos(\theta+\phi) = r\cos\theta\cos\phi - r\sin\theta\sin\phi$$

$$y' = r\sin(\theta+\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$$

$$x' = x\cos\theta - y\sin\theta$$

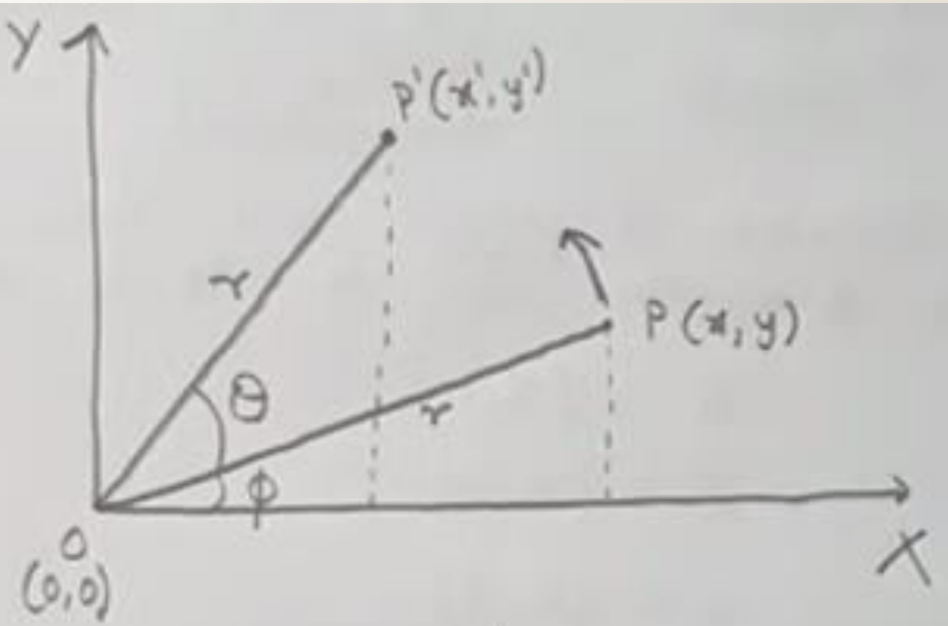
$$y' = x\sin\theta + y\cos\theta$$



2D Rotation

$$x' = r\cos(\theta + \phi)$$
$$= r(\cos\theta\cos\phi - \sin\theta\sin\phi)$$
$$= r\cos\theta\cos\phi - r\sin\theta\sin\phi$$
$$= x\cos\theta - y\sin\theta$$

$$y' = r\sin(\theta + \phi)$$
$$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$= r\sin\theta\cos\phi + r\cos\theta\sin\phi$$
$$= x\sin\theta + y\cos\theta$$



$$x = r\cos\phi$$
$$y = r\sin\phi$$

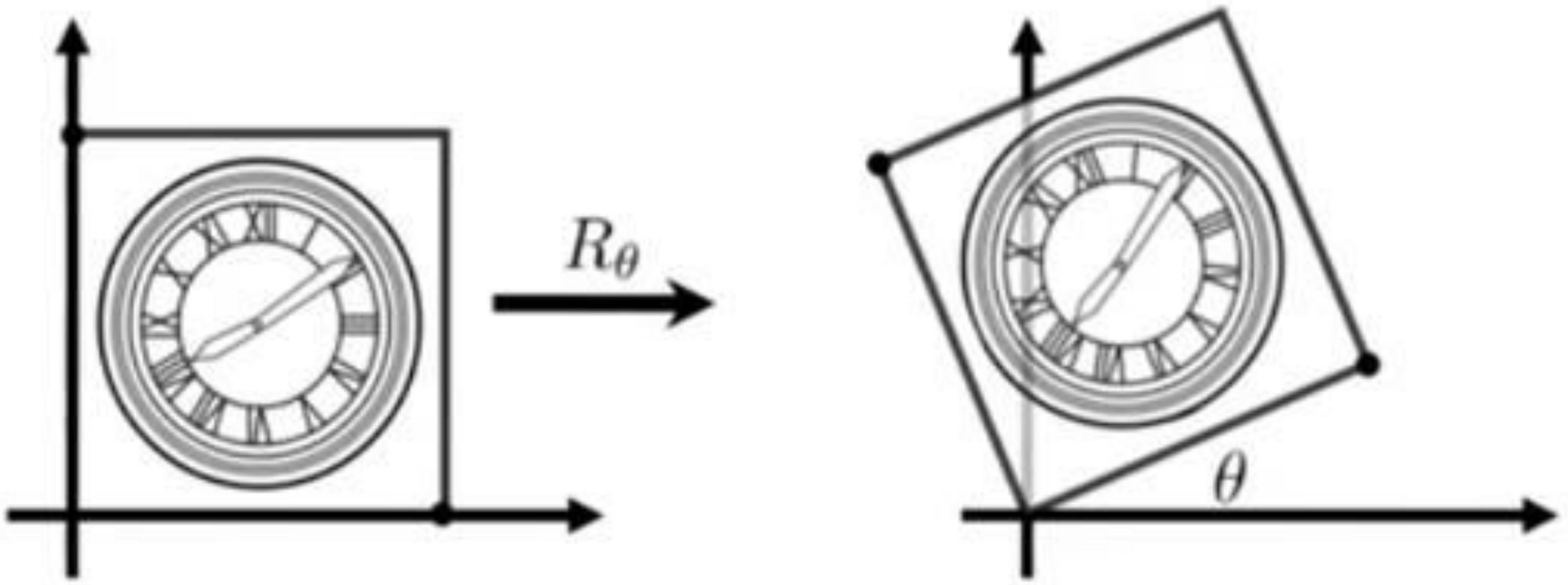
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

Matrix Representation : → Rotation matrix

$$P' = R\,P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation Matrix



$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2D Rotation Example 1

– Find the transformed point, P', caused by rotating P= (5, 1) about the origin through an angle of 90°.

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \bullet \begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}x\cdot\cos\theta - y\cdot\sin\theta \\ x\cdot\sin\theta + y\cdot\cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}5\cdot\cos 90 - 1\cdot\sin 90 \\ 5\cdot\sin 90 + 1\cdot\cos 90\end{bmatrix}$$

$$= \begin{bmatrix}5\cdot 0 - 1\cdot 1 \\ 5\cdot 1 + 1\cdot 0\end{bmatrix}$$

$$= \begin{bmatrix}-1 \\ 5\end{bmatrix}$$

2D Rotation Example 2

A point (4,3) is rotated in counter clockwise direction by the angle of 45°. Find the rotation matrix R and the resultant points

$$P' = R\,P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}_{2\times2} \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{2\times1}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} + \frac{3}{\sqrt{2}} \end{bmatrix}_{2\times1}$$

2D Rotation Example

Q. Rotate a Δ^{le} $A(0,0)$ $B(2,2)$ $C(4,2)$ about the origin by an angle of 45°.

$$A' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4\sqrt{2} \end{bmatrix}$$

$$C' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}$$

2D rotation Example 3

One triangle is given (2,2) (8,2) (5,5).

Rotate the triangle 90°.

here $\theta = 90^\circ$

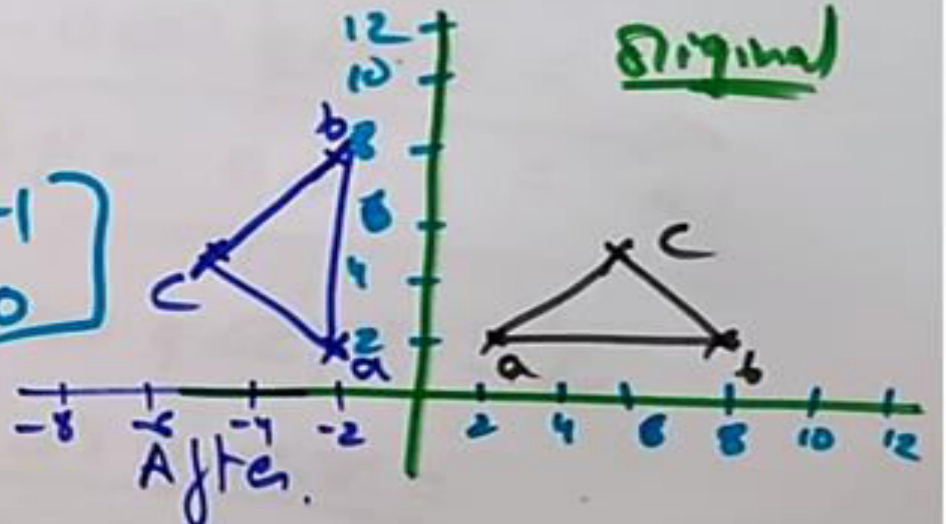
Answer

One triangle is given (2,2) (8,2) (5,5).

Rotate the triangle 90°.

here $\theta = 90^\circ$

$$R = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

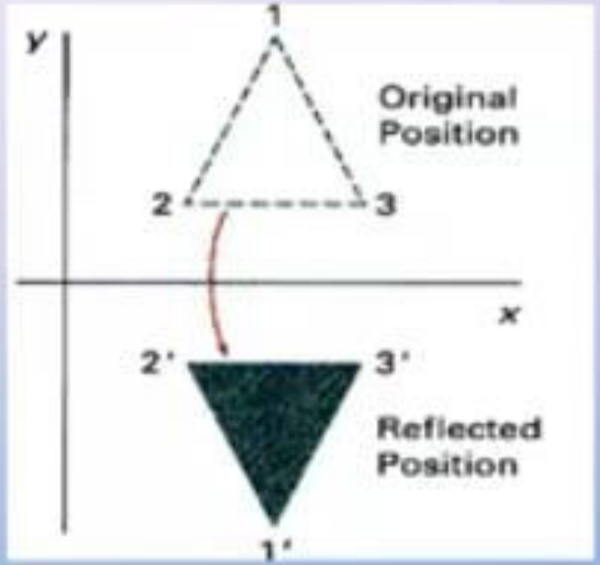
$$C' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

2D Reflection

2D Reflection

- Transformation that produces a mirror image of an object is called *reflection*.

- Image is generated relative to an axis of reflection by rotating the object **180°** about the reflection axis.

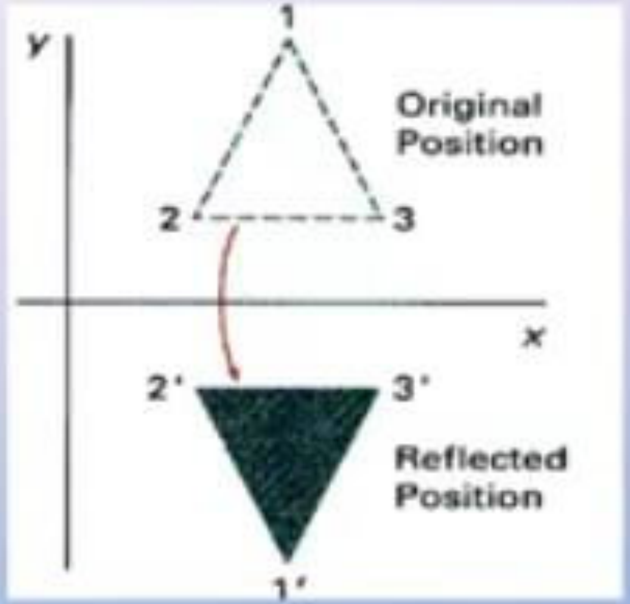


2D Reflection

A Common reflections as follows:

1. Reflection about the line y=0 (the x-axis) is accomplished with the transformation matrix:

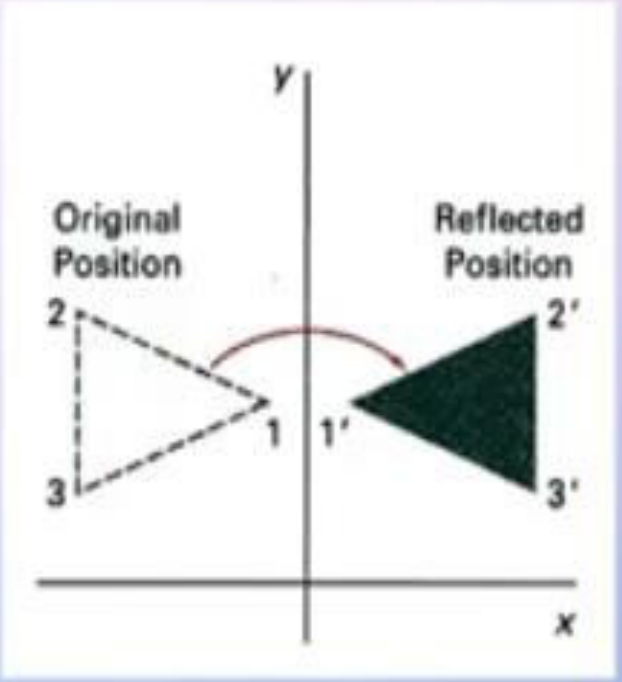
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \text{Reflection in the x-axis}$$



2D Reflection

2. Reflection about the line x=0 (the y-axis) is accomplished with the transformation matrix

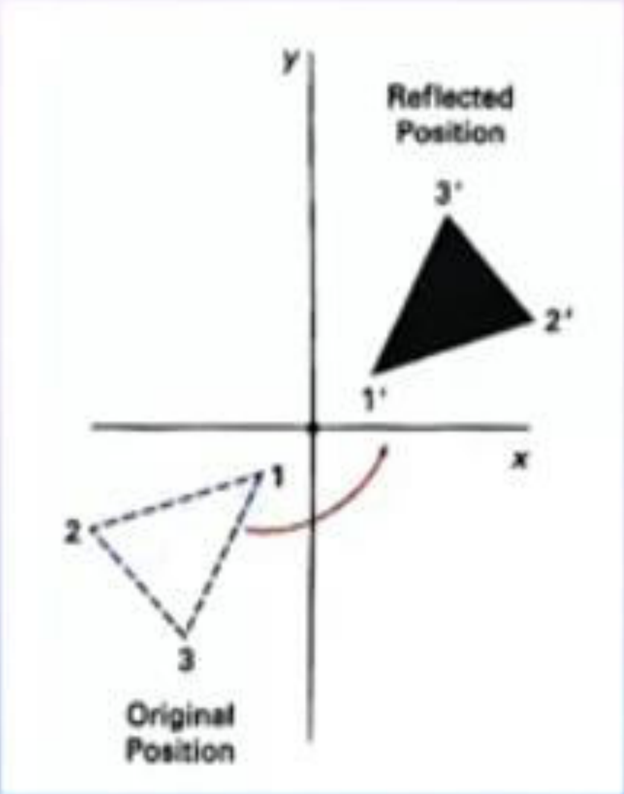
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{Reflection in the y-axis}$$



2D Reflection

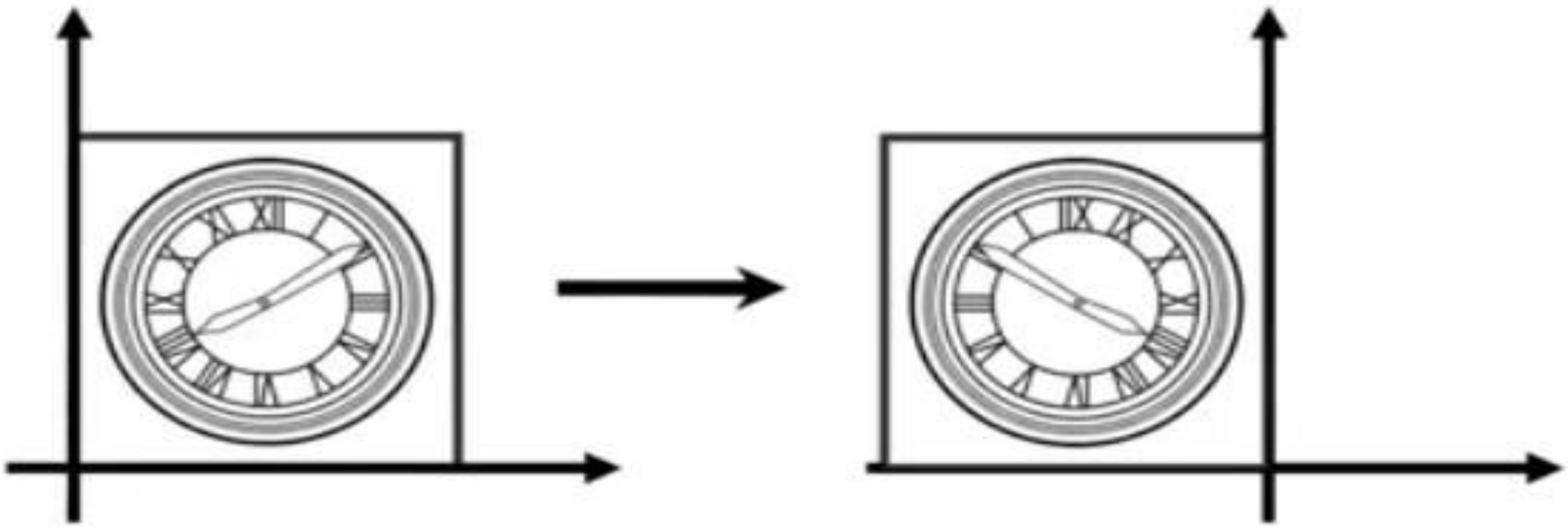
3.Reflection about the origin which is equivalent to the rotation matrix R(θ) with θ=180°

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \text{Reflection about the origin}$$



2D Reflection

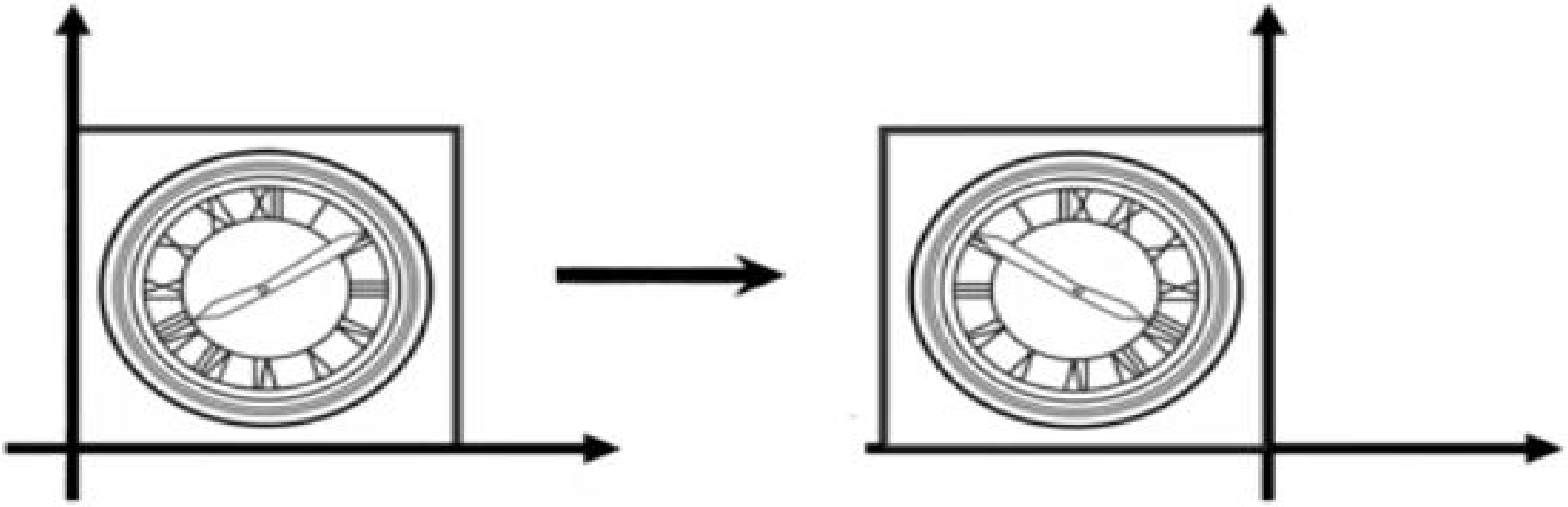
Reflection



$$x' = -x$$
$$y' = y$$

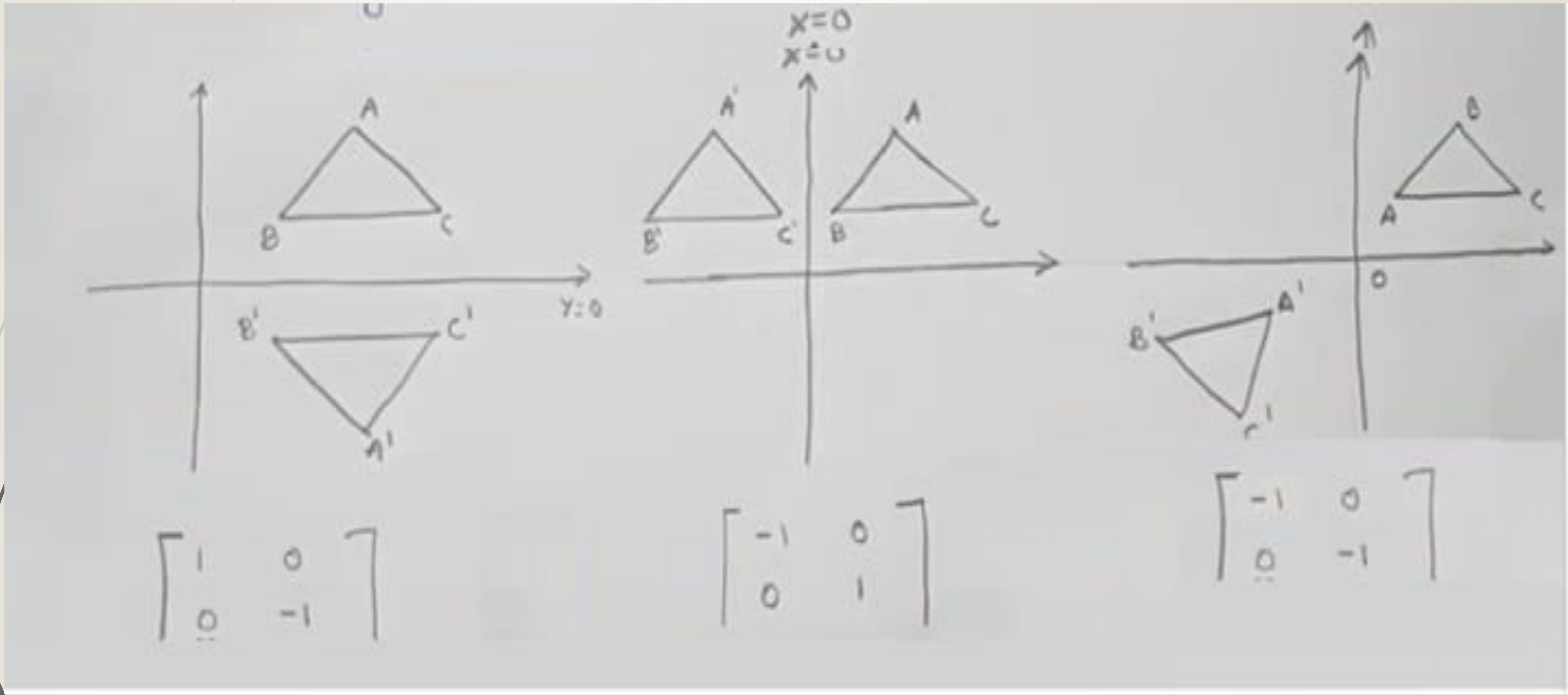
2D Reflection

Reflection Matrix

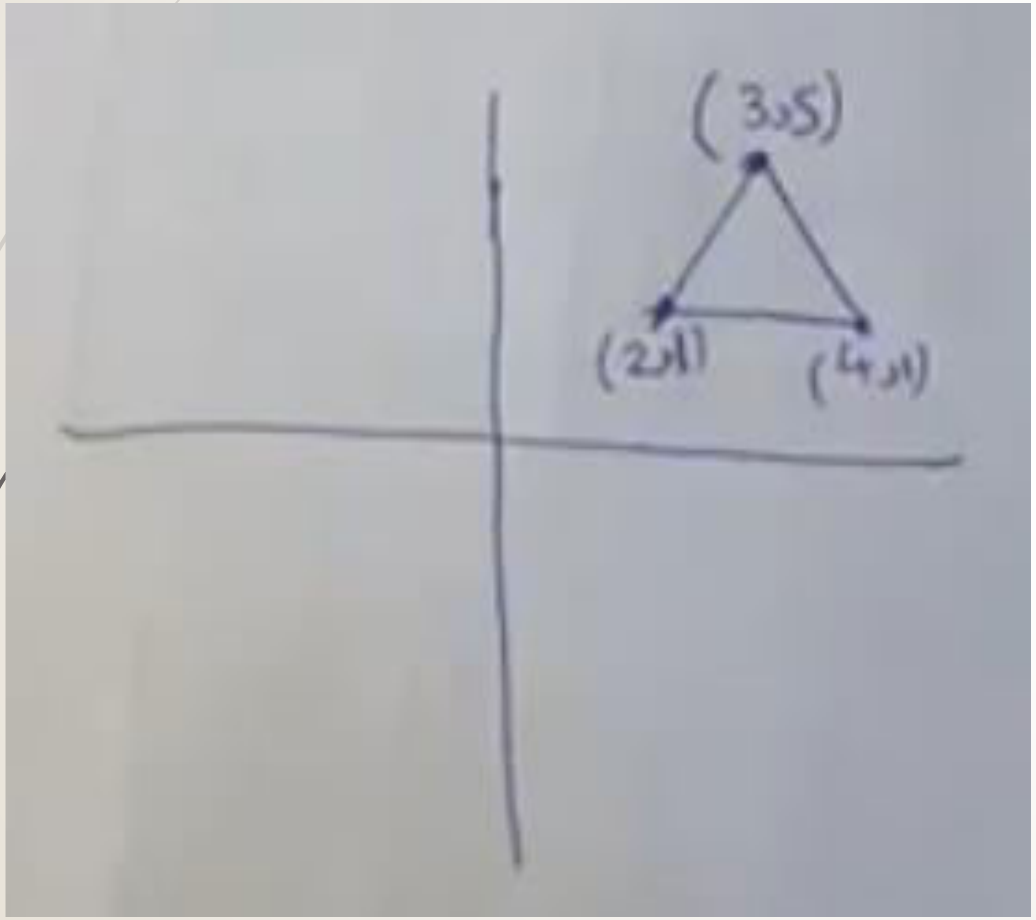


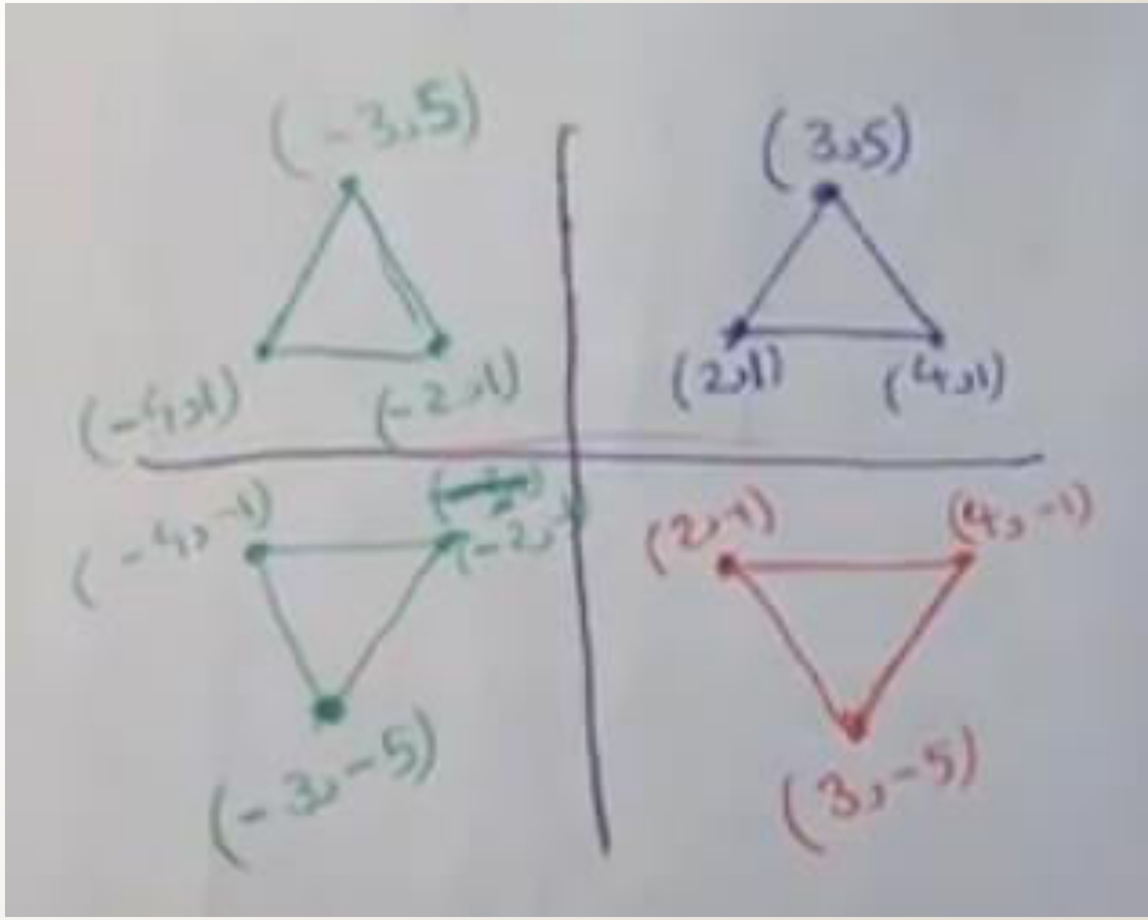
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Reflection



2D Reflection Example 1





2D Reflection Example 2

Example1: Given a triangle with coordinate points A(3, 4), B(6, 4), C(5, 6). Apply the reflection transformation:

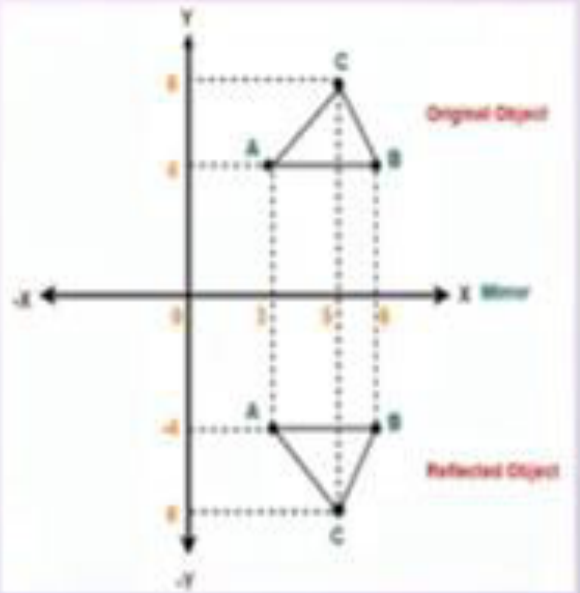
1- about the x-axis

2- about the y-axis

Sol: \ 1- about the x-axis

$$y=0 \quad : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \text{reflection matrix}$$

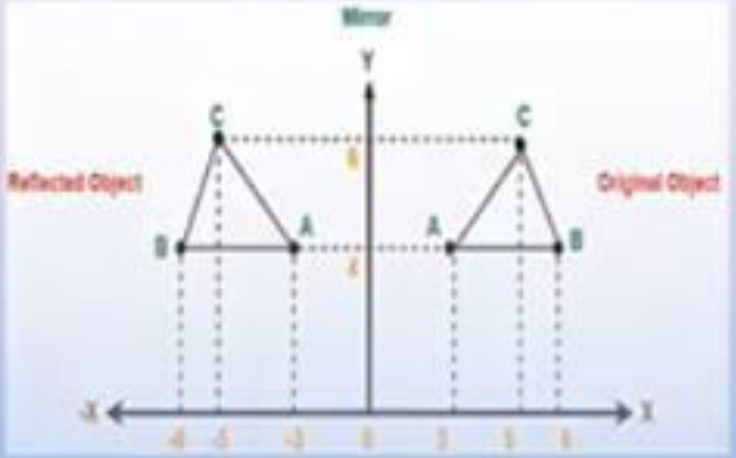
$$\begin{bmatrix} 3 & 4 \\ 6 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 6 & -4 \\ 5 & -6 \end{bmatrix}$$



Sol: \ 2- about the y-axis

$$x=0 \quad : \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{reflection matrix}$$

$$\begin{bmatrix} 3 & 4 \\ 6 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -6 & 4 \\ -5 & 6 \end{bmatrix}$$

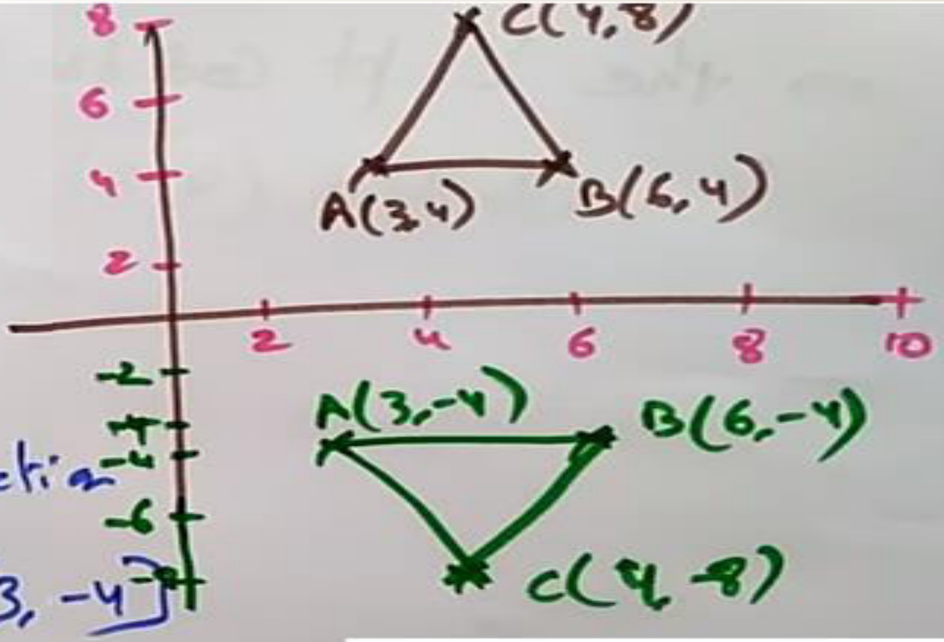


2D Reflection Example 3

A(3,4) B(6,4) C(4,8)

The matrix for Re(x) is

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



The 'A' pt coordinates after reflection

$$(x,y) = (3,4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [3,-4]$$

The 'B' pt coordinates after reflection

$$(x,y) = (6,4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [6,-4]$$

the 'c' pt coordinates after reflection

$$(x,y) = (4,8)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [4,-8]$$

2D Reflection Example 4

Example2: Given a triangle with coordinate points A(3, 4), B(6, 4), C(5, 6). Apply the reflection transformation:

1- about the origin

2- about the diagonal line y=x

3- about the diagonal line y=-x

Sol: \ 1- about the origin

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow$ reflection matrix

$$\begin{bmatrix} 3 & 4 \\ 6 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -6 & -4 \\ -5 & -6 \end{bmatrix}$$

Sol: \ 2- about the diagonal line y=x

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow$ reflection matrix

$$\begin{bmatrix} 3 & 4 \\ 6 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 4 & 6 \\ 6 & 5 \end{bmatrix}$$

Sol: \ 3- about the diagonal line y=-x

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow$ reflection matrix

$$\begin{bmatrix} 3 & 4 \\ 6 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ -4 & -6 \\ -6 & -5 \end{bmatrix}$$

2D Shearing

2D Shearing

- Distorting or changing the shape of an object by differentially moving some of its vertices as if the object internal layers are sided over each other is called **Shear.**

- Shears either shift coordinates x values or y values.

- Similar to scaling, the shear transformation requires two parameters (s_x, s_y) not on the main diagonal of the transformation matrix but on the other two positions.

$$\begin{bmatrix} 1 & S_x \\ S_y & 1 \end{bmatrix}$$

2D Shearing

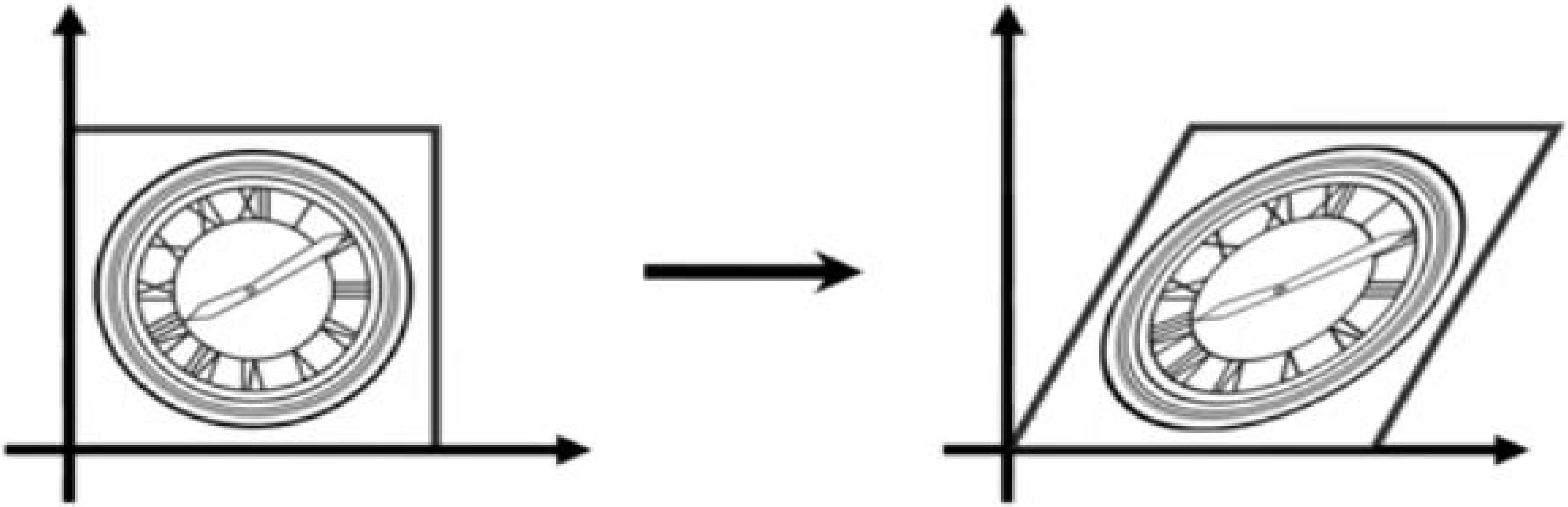
- Applying a shear transformation sh(sx, sy) to point (x,y) yields the point ($\bar{x}$, $\bar{y}$) with the new coordinates.

$$[\bar{x} \quad \bar{y}] = [x \quad y]\begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix} = [x + y * s_y \quad x * s_x + y]$$

- Shear transformations are carried out with respect to the origin of the coordinate system so that an object that is not centered around the origin will not only be deformed by a shear transformation but also shifted.

- If s_x =0 the shear take place in the x direction and in the y direction if s_y =0

Shear Matrix



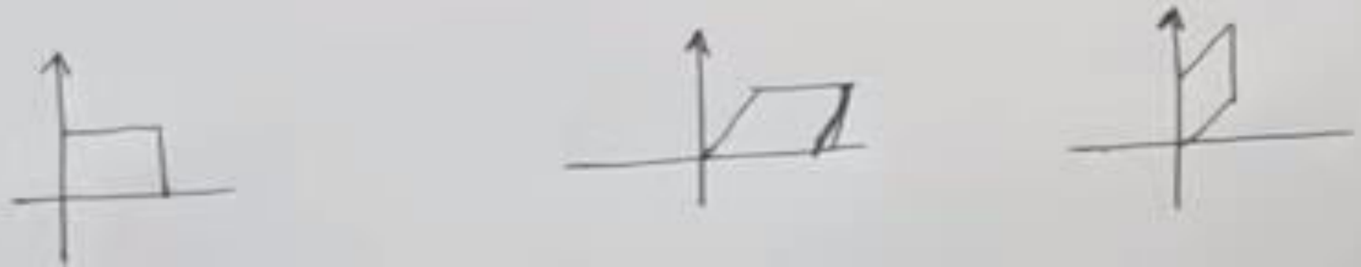
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shearing : → The shearing transformation distorts the shape of the object
↓
Skewing
→ There are 2 types of shearing transformation:

(i) X - shearing
(ii) Y - shearing

(i) **X-shearing** : → It preserves the y coordinates but changes the x values.
→ For any point P(x,y)
P'(x+shx·y , y), where shx is the shearing vector in x direct[ion]

(ii) **Y-shearing** : → The x coordinate remains same while there is a change in y value.
→ For any point P(x,y)
P'(x, y+shy·x), where shy is the shearing vector in y direction



2D Shear Example 2

Example: Consider the square whose vertices are: A(0,0), B(1,0), C(1,1), and D(0,1). Perform the shear transformation:

1- In the x direction
2- In the y direction
3- In both direction

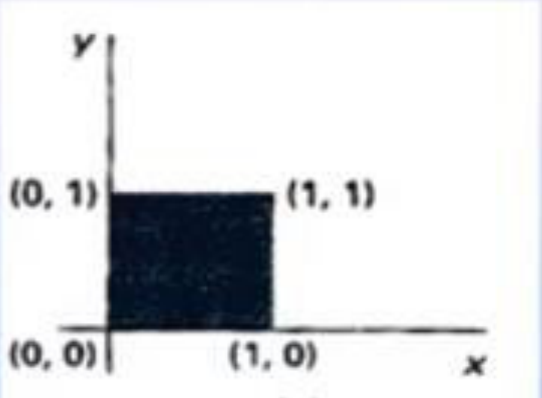
Where s_x=1 and s_y=2

Sol: \ 1. Shear in the x direction: $\begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$

$s_x=0$: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ → shear matrix

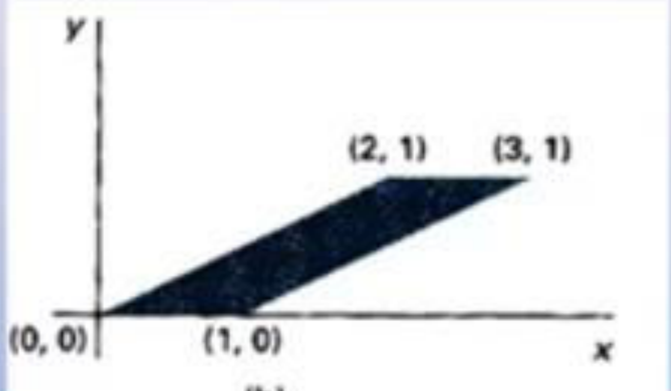
$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}$$

Befor



(a)

After



(b)

2D Shear Example 2

<u>Example:</u> Consider the square whose vertices are: A(0,0), B(1,0), C(1,1), and D(0,1). Perform the shear transformation:

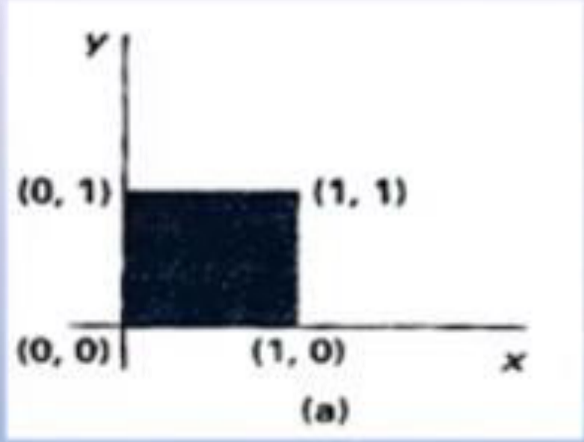
1- In the x direction
2- In the y direction
3- In both direction

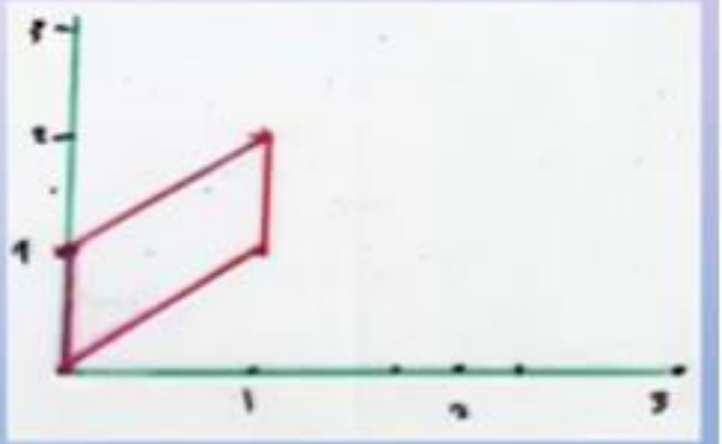
Where s_x=1 and s_y=2

Sol: \ 2. Shear in the y direction: $\begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$

$s_y=0$; $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ → shear matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$





2D Shear Example 2

Example: Consider the square whose vertices are: A(0,0), B(1,0), C(1,1), and D(0,1). Perform the shear transformation:

1- In the x direction

2- In the y direction

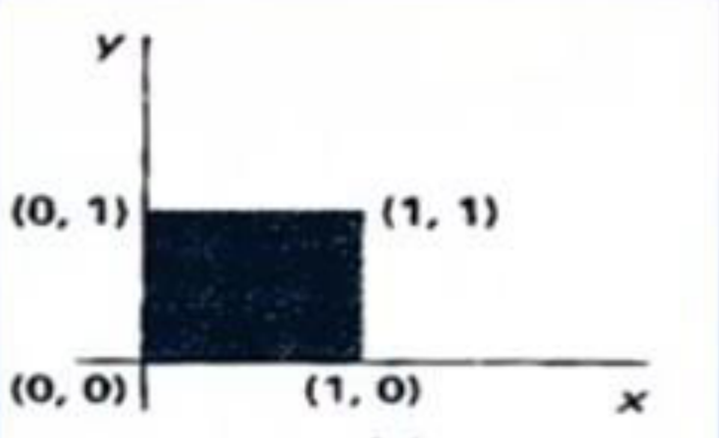
3- In both direction

Where s_x=1 and s_y=2

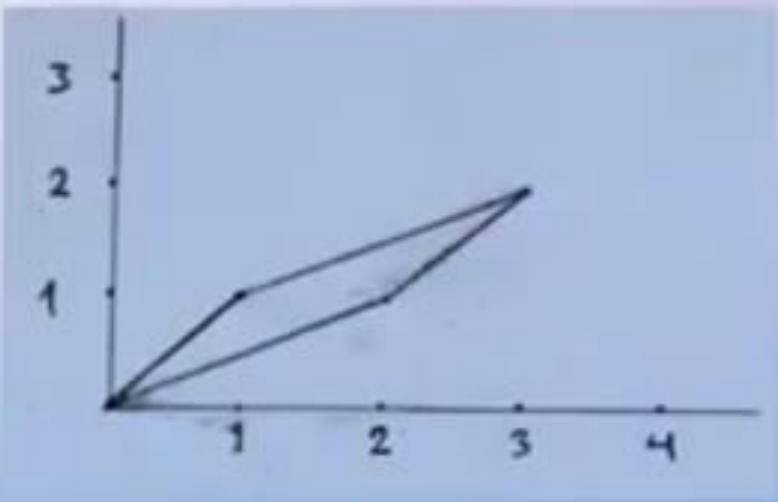
Sol: \ 3- In both direction $\begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$

s_x=1 ; s_y=2 ; $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \rightarrow$ shear matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$$



(a)



2D Shear Example 1

A triangle with (2,2) (0,0) & (2,0). Apply shearing factor 2 on x-axis and 2 on y-axis. Find out the new coordinates of triangle

Solution:

A(2,2) B(0,0) C(2,0) $Sh_x = 2$; $Sh_y = 2$

x-Shear:

$A = (x_1, y_1)$ $A(x_0\ y_0) = (2,2)$

$x_1 = x_0 + sh_x \cdot y_0 = 2 + 2 * 2 = 6$ (6,2)

$y_1 = y_0 = 2$

$B = (0,0)$ (x_0, y_0) $B = (x_1, y_1)$

$x_1 = x_0 + sh_x \cdot y_0 = 0 + 2 * 0 = 0$ (0,0)

$y = y_0 = 0$

$C = (x_0\ y_0) = (2,0)$

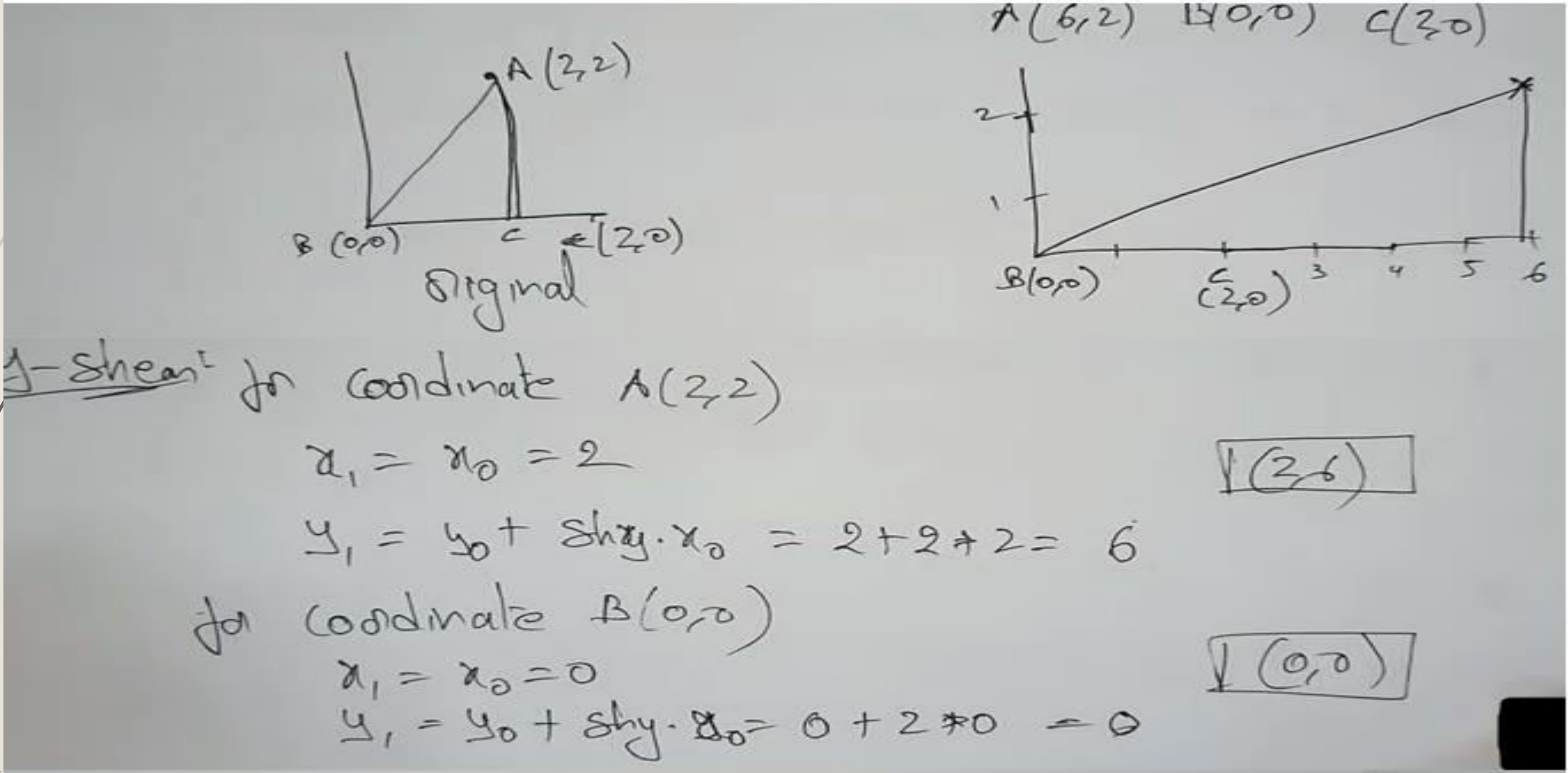
$C = (x_1, y_1)$

$x_1 = x_0 + Sh_x \cdot y_0 = 2 + 2 * 0 = 2$

$y_1 = y_0 = 0$

(2,0)

2D Shear Example 1



2D Shear Example 1